CRAMER RAO BOUND COMPUTATION FOR VELOCITY ESTIMATION IN THE BROAD-BAND CASE USING THE MELLIN TRANSFORM*

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ABSTRACT

We present explicit expressions of the Cramer Rao Bounds (CRB) for estimating the velocity and the radial position of a target illuminated by a signal which does not satisfy Woodward's conditions (small relative bandwidth and reasonable time-bandwidth product). Classically, the variances lower bounds of estimates are obtained by inverting the Fisher Information Matrix (FIM) built according to the classical Maximum Likelihood (ML) estimation theory. The direct application of this procedure to broad-band situations has so far not been performed for technical raisons. We show that the use of the Mellin Transform allows a simple computation of the FIM and leads to a clear interpretation of its elements in the time-frequency plane.

1. INTRODUCTION

In radar or sonar, estimating the parameters such as the velocity or the position of a target is often a delicate problem. Consider an analytic signal $z(t)$ emitted on a moving target. The received signal is

$$x(t,\theta) = A_0 T_\theta z(t)\, e^{i\phi} + b(t) \tag{1}$$

where T_θ is a transformation acting on the signal $z(t)$ with a vector θ of unknown parameters (delay, Doppler shift, Doppler compression, etc..), where A_0 is the amplitude, ϕ a phase change and b(t) a zero-mean white gaussian noise with σ^2 variance. When the probability density of the parameters A_0 and ϕ is unknown, the ML ratio Λ to maximize, according to the Maximum Likelihood estimation theory, is given by the square modulus of the cross-ambiguity function:

$$\Lambda(\theta,\theta_0) = \frac{1}{2\sigma^2}\,\frac{\left|\int_{-\infty}^{+\infty} x(t,\theta)\, T^*_{\theta_0} z(t)\, dt\right|^2}{\int_{-\infty}^{+\infty} \left|T_{\theta_0} z(t)\right|^2\, dt} \tag{2}$$

The efficiency of an estimator $\hat\theta$ is generally measured by its variance $Var(\theta-\hat\theta)$. For an unbiased estimator ($E(\hat\theta) = \theta$), this variance has a lower value given by the CRB [1]. The CRB are obtained by inverting the FIM defined as:

$$J_{i,j} = \left(-E\left[\frac{\partial^2\Lambda}{\partial\theta_i\partial\theta_j}\right]\right)_{i,j} \tag{3}$$

where θ_i denotes each component of the vector θ.

1.1 The Narrow-Band Case

Under Woodward's conditions [2], the Doppler effect can be approximated by a shift in frequency of the signal z(t). Hence, the received signal $x(t,\theta)$ can be put in the form:

$$x(t,\theta) = A_0\, z(t-\tau)\, e^{2i\pi\nu t}\, e^{i\phi} + b(t) \tag{4}$$

where $\nu = 2vf_0/c$ is the Doppler shift and τ the delay (radial position $c\tau/2$, c propagation velocity, f_0 center frequency). In this case the FIM (3) can be easily calculated and leads to

$$J = \frac{4\pi^2 A_0^2}{\sigma^2}\begin{pmatrix} \sigma_f^2 & f_0\, t_0 - m \\ f_0\, t_0 - m & \sigma_t^2 \end{pmatrix} \tag{5}$$

where the first order moments f_0 and t_0 represent the mean frequency and the mean epoch, where the second order moments σ_f and σ_t represent the bandwidth and the duration of the signal. The parameter m is the modulation index of the signal. Each lower bound of the variance of estimates are obtained by inverting the matrix (5). These well known results prove that the best signal in radar (good range and velocity resolutions) is characterized by a high time-bandwidth product.

1.2 The Broad-Band Case

In that case, the problem of estimating a velocity does not consist in estimating a Doppler shift but a Doppler compression factor. Thus, the echo $x(t)$ can be put in the form:

$$x(t) = A_0\, z(a_0^{-1}t - b_0)\, e^{i\phi} + b(t) \tag{6}$$

The statistic to maximize is given by the square modulus of the broad-band cross-ambiguity function which is rewritten in the frequency domain:

$$\Lambda = \frac{a}{2\sigma^2}\left|\int_0^{+\infty} X(f)\, Z^*(af)\, e^{2i\pi abf}\, df\right|^2 \tag{7}$$

where the parameters $a = (c+v)/(c-v)$ and b represent the Doppler compression and the delay parameters to estimate. The direct calculation of the FIM (3) by classical methods is not very easy and does not lead to a simple interpretation as in the narrow-band case (5). In the next section and using the Mellin transform [3, 4] already used in Broad-Band signal analysis (Affine Time-Frequency Distributions [5], Wavelet Transform [6], Broad-Band Ambiguity Functions [7]), the FIM computation is, on the one hand, easily performed and leads, on the other hand, to a physical interpretation of its coefficients.

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2. THE FISHER INFORMATION MATRIX IN BROAD-BAND CASE

2.1 The Mellin Transform

The Mellin transform which plays an important role in the computation and the physical interpretation of the FIM's coefficients has been well defined in [3] and acts on the analytic signal $Z(f)$ in frequency by:

$$M^{\xi}[Z](\beta) = \int_0^{+\infty} Z(f)\, e^{2i\pi\xi f}\, f^{2i\pi\beta + r}\, df \tag{8}$$

This transform can be interpreted as the coefficient of the decomposition of the signal onto a hyperbolic signals basis with a group delay law given by the equation $t = \xi + \beta/f$ with the invariant scalar product given by:

$$\int_0^{+\infty} Z_1(f) Z_2^*(f) f^{2r+1} df = \int_{-\infty}^{+\infty} M^{\xi}[Z_1](\beta) M^{\xi *}[Z_2](\beta) d\beta \tag{9}$$

The dual Mellin variable β therefore characterizes the coefficient of an hyperbola in the time-frequency half plane. The parameter r is free but is chosen here equal to $-1/2$ to preserve the classical scalar product. In the following, the ξ parameter will be equal to zero (signal centered around the mean epoch $\xi = 0$) and the transform will be noted $M[Z](\beta)$. Using an a priori knowledge of the localization of the signal in the time-frequency half plane (bandwidth, relative bandwidth, duration), it is now possible to perfectly determine the spread of the signal in the Mellin space.

The main property of the Mellin transform is the property of scale invariance:

$$\begin{array}{ccc} Z(f) & \longrightarrow & Z'(f) = \sqrt{a}\, Z(af) \\ \downarrow & & \downarrow \\ M[Z](\beta) & \longrightarrow & M[Z'](\beta) = a^{-2i\pi\beta} M[Z](\beta) \end{array} \tag{10}$$

which is useful when rewriting (7):

$$\Lambda = \frac{1}{2\sigma^2} \left| \int_{-\infty}^{+\infty} M[X](\beta)\, M^*[Z_b](\beta)\, a^{2i\pi\beta}\, d\beta \right|^2 \tag{11}$$

with $Z_b(f) = Z(f)\exp(2i\pi bf)$. Another important property of the Mellin transform, useful for computation of the FIM coefficients, is the diagonalization of the operator B defined by

$$BZ(f) = -\frac{1}{2i\pi} \left(f \frac{d}{df} + \frac{1}{2} \right) Z(f) \tag{12}$$

which is transformed as $M[BZ](\beta) = \beta M[Z](\beta)$

2.2 Broad-Band Expression of the Fisher Information Matrix

Proposition: *The Fisher Information Matrix has the form* [4] :

$$J = \frac{4\pi^2 A_0^2}{\sigma^2} \begin{pmatrix} \sigma_\beta^2 & f_0\beta_0 - M \\ f_0\beta_0 - M & \sigma_f^2 \end{pmatrix} \tag{13}$$

where the parameters σ_f and f_0 define the bandwidth and the mean frequency of the signal and where the parameters β_0, σ_β are given by :

$$\beta_0 = \int_{-\infty}^{+\infty} \beta\, |M[Z](\beta)|^2\, d\beta \tag{14}$$

$$\sigma_\beta^2 = \int_{-\infty}^{+\infty} (\beta - \beta_0)^2\, |M[Z](\beta)|^2\, d\beta \tag{15}$$

The first and second order moments can be viewed respectively as the mean β and the spread of the signal Z in Mellin space. The broad-band modulation index M defined by

$$M = -\frac{1}{2\pi} Im \int_0^{+\infty} f^2 \frac{dZ}{df} Z^*(f)\, df \tag{16}$$

plays the same role for the hyperbolic signals as the narrow-band modulation index m for the 'chirp' signals. Finally, the ratio A_0/σ is the Signal-to-Noise Ratio.

To estimate the quality of the compression and delay parameters, the FIM must be inverted. Each term of the inverse matrix J^{-1} gives the variance lower bound of each estimator. As the estimators are unbiased and efficient (high SNR and measures), the CRB are reached and we obtain the following important new results:

- The variance of the delay estimator $\hat{b}$ is given by:

$$\begin{aligned} E\left[(b - \hat{b})^2\right] &= \frac{\sigma^2}{4\pi^2 A_0^2} \frac{\sigma_\beta^2}{\sigma_f^2 \sigma_\beta^2 - (M - \beta_0 f_0)^2} \\ &\geq \frac{\sigma^2}{4\pi^2 A_0^2} \frac{1}{\sigma_f^2} \end{aligned} \tag{17}$$

This first result (17) shows that the delay (or range) resolution is always related to the inverse of the signal spread in frequency as in the narrow-band case.

- The variance of the compression estimator $\hat{a}$ is given by:

$$\begin{aligned} E\left[(a - \hat{a})^2\right] &= \frac{\sigma^2}{4\pi^2 A_0^2} \frac{\sigma_f^2}{\sigma_f^2 \sigma_\beta^2 - (M - \beta_0 f_0)^2} \\ &\geq \frac{\sigma^2}{4\pi^2 A_0^2} \frac{1}{\sigma_\beta^2} \end{aligned} \tag{18}$$

- The variance of the velocity estimator $\hat{v}$ is given by:

$$E\left[(v - \hat{v})^2\right] = \frac{c^2}{4} E\left[(a - \hat{a})^2\right] \tag{19}$$

These two results (18) and (19) are very important because they prove that the compression (or velocity) resolution depends only on the inverse of the signal spread in the Mellin space instead of the signal duration as in the narrow-band case.

- The covariance of the cross-estimators is given by:

$$E\left[(a-\hat{a})(b-\hat{b})\right] = \frac{\sigma^2}{4\pi^2 A_0^2}\,\frac{M-\beta_0 f_0}{\sigma_f^2\sigma_\beta^2-(M-\beta_0 f_0)^2} \qquad (20)$$

Under Woodward's assumptions (narrow-band case, $v/c \ll 1$), the hyperbolas which delimit the signal in the time-frequency half plane, may be replaced by straight lines parallel to the frequency axis. In that case, the parameter β, β_0, σ_β, M and $a=(c+v)/(c-v)$ can be respectively approximated by $f_0 t$, $f_0 t_0$, $f_0\sigma_t$, $f_0 m$ and $1+2v/c$. Substituting these approximations in (18) and (19), we obtain the classical narrow-band results:

$$E\left[(v-\hat{v})^2\right] = \frac{c^2\sigma^2}{16\pi^2 A_0^2 f_0^2}\,\frac{\sigma_f^2}{\sigma_t^2\sigma_f^2-(m-f_0 t_0)^2} \qquad (21)$$

2.2 Proof of the Proposition

The main idea for FIM derivation is to compute the FIM coefficients from the statistics Λ (11) rewritten in Mellin space rather than a direct computation. To simplify the demonstration, all the partial derivatives of the statistic Λ with respect to parameters a and b will be evaluated at the point $\mathrm{O}(a=a_0=1, b=b_0=0)$.

If we note $A(a,b)$, the classical cross-ambiguity function rewritten in Mellin space:

$$A(a,b) = \int_{-\infty}^{+\infty} M[X](\beta)\, M^*[Z_b](\beta)\, a^{2i\pi\beta}\, d\beta \qquad (22)$$

all the partial derivatives of A with respect to parameters a and b and evaluated at the point $\mathrm{O}(a=1, b=0)$, when using the property of unitarity of the Mellin transform (9), lead to:

$$\frac{\partial A}{\partial a} = 2i\pi \int_{-\infty}^{+\infty} \beta\, M[X](\beta)\, M^*[Z](\beta)\, d\beta \qquad (23)$$

$$\frac{\partial^2 A}{\partial a^2} = 2i\pi \int_{-\infty}^{+\infty} \beta(2i\pi\beta-1)\, M[X](\beta)\, M^*[Z](\beta)\, d\beta \qquad (24)$$

$$\begin{aligned}\frac{\partial A}{\partial b} &= 2i\pi \int_{-\infty}^{+\infty} M[X](\beta)\, M^*[fZ(f)](\beta)\, d\beta &(25)\\ &= 2i\pi \int_{0}^{+\infty} f\, X(f)\, Z^*(f)\, df &(26)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 A}{\partial a \partial b} &= 2i\pi \int_{-\infty}^{+\infty} M[X](\beta)\, M^*[fZ(f)](\beta)\, d\beta \\ &- 4\pi^2 \int_{-\infty}^{+\infty} \beta\, M[X](\beta)\, M^*[fZ(f)](\beta)\, d\beta \quad (27)\end{aligned}$$

$$\frac{\partial^2 A}{\partial b^2} = -4\pi^2 \int_{0}^{+\infty} f^2\, X(f)\, Z^*(f)\, df \qquad (28)$$

The first coefficient J_{11} of the FIM takes the form:

$$J_{11} = -\frac{1}{\sigma^2}\, E\left[Re\left(A^*\,\frac{\partial^2 A}{\partial a^2} + \left|\frac{\partial A}{\partial a}\right|^2\right)\right] \qquad (29)$$

Using relations (23) and (24), J_{11} becomes:

$$\begin{aligned} J_{11} = -\frac{1}{\sigma^2} Re \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} & E\left[M[X](\beta_1) M^*[X](\beta_2)\right] \\ & \times M[Z](\beta_1) M^*[Z](\beta_2) \\ & \times \left[2i\pi\beta_1(2i\pi\beta_1-1)+4\pi^2\beta_1\beta_2\right] d\beta_1\, d\beta_2 \qquad (30)\end{aligned}$$

The noise $b(t)$ is a zero mean white gaussian noise. If we note $C(\beta_1-\beta_2)$ the covariance of the Mellin transform of X, it is easy to show that:

$$C(\beta_1-\beta_2) = E\left[M[X](\beta_1) M^*[X](\beta_2)\right] \qquad (31)$$

$$= A_0^2\, M[Z](\beta_1) M^*[Z](\beta_2) + \sigma^2\, \delta(\beta_1-\beta_2) \qquad (32)$$

while the covariance $C(f_1-f_2)$ of the Fourier transform of the signal is given by:

$$\begin{aligned} C(f_1-f_2) &= E\left[X(f_1) X^*(f_2)\right] &(33)\\ &= A_0^2\, Z(f_1)\, Z^*(f_2) + \sigma^2\, \delta(f_1-f_2) &(34)\end{aligned}$$

When substituting the relation (32) in (30), we obtain the expression of the J_{11} coefficient:

$$\begin{aligned} J_{11} &= \frac{4\pi^2 A_0^2}{\sigma^2} \int_{-\infty}^{+\infty} (\beta-\beta_0)^2\, |M[Z](\beta)|^2\, d\beta &(35)\\ &= \frac{4\pi^2 A_0^2}{\sigma^2}\, \sigma_\beta^2 &(36)\end{aligned}$$

The computation of the J_{22} coefficient of the FIM is easily performed by:

$$J_{22} = -\frac{1}{\sigma^2}\, E\left[Re\left(A^*\,\frac{\partial^2 A}{\partial b^2} + \left|\frac{\partial A}{\partial b}\right|^2\right)\right] \qquad (37)$$

and, using relations (26) and (28), leads to:

$$\begin{aligned} J_{22} &= \frac{4\pi^2}{\sigma^2}\, Re \int_{0}^{+\infty}\int_{0}^{+\infty} C(f_1-f_2) \\ &\times\ Z^*(f_1)\, Z(f_2)\, (f_1^2 - f_1 f_2)\, df_1\, df_2 \qquad (38)\end{aligned}$$

Substituting in (38) the covariance given by (34), we obtain the J_{22} coefficient proposed in (13).

Finally, the last symmetrical coefficient J_{12} or J_{21} computation is given by:

$$J_{12} = -\frac{1}{\sigma^2}\, E\left[Re\left(X^*\,\frac{\partial^2 X}{\partial a\,\partial b} + \frac{\partial X}{\partial a}\,\frac{\partial X^*}{\partial b}\right)\right] \qquad (39)$$

which, using relations (23), (26) and (27), leads to:

$$J_{12} = -\frac{4\pi^2}{\sigma^2} Re \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\beta_1 - \beta_2)\, C(\beta_1 - \beta_2) \times M^*[f Z(f)](\beta_1)\, M[Z](\beta_2)\, d\beta_1\, d\beta_2 \quad (40)$$

Substituting in (40) the covariance given in (32), the relation (40) becomes:

$$J_{12} = -\frac{4\pi^2 A_0^2}{\sigma^2} Re \int_{-\infty}^{+\infty} \beta_1 M[Z](\beta_1)\, M^*[f Z(f)](\beta_1)\, d\beta_1$$

$$+\frac{4\pi^2 A_0^2}{\sigma^2} Re \int_{-\infty}^{+\infty} M[Z](\beta_1)\, M^*[f Z(f)](\beta_1)\, d\beta_1 \times \int_{-\infty}^{+\infty} \beta_2\, |M[Z](\beta_2)|^2\, d\beta_2 \quad (41)$$

This last expression can be easily transformed in the frequency domain using the unitarity property (9) of the Mellin transform and using the operator B defined by equation (12). Hence, J_{12} is rewritten as:

$$\begin{aligned} J_{12} &= -\frac{4\pi^2 A_0^2}{\sigma^2} \left[Re \int_0^{+\infty} B Z(f)\, f\, Z^*(f)\, df \right. \\ &+ \left. \beta_0 \int_0^{+\infty} f\, |Z(f)|^2\, df \right] \\ &= -\frac{2\pi^2 A_0^2}{\sigma^2} Re \int_0^{+\infty} i f^2 \frac{dZ}{df} Z^*(f)\, df \\ &+ \frac{4\pi^2 A_0^2}{\sigma^2} \beta_0 f_0 \end{aligned} \quad (42)$$

With the definition of the broad-band modulation index M defined by (16), the coefficient J_{12} is finally derived.

3. CONSTRUCTION OF OPTIMAL SIGNAL

Consider a monochromatic and analytic signal given by its equation $Z(f) = \delta(f - f_0)$. This signal has a Mellin transform given by $M^\xi[Z](\beta) = f_0^{2i\pi\beta + r} \exp(2i\pi\xi f_0)$. We can therefore perfectly determine the frequency law of the signal Z(f) as the function of the Mellin variable:

$$f_0 = \exp\left(\frac{1}{2\pi} \frac{d\phi}{d\beta}\right) \quad (43)$$

where $\phi(\beta)$ is the phase of the Mellin transform of Z. Extending this relation, we obtain the expression of the frequency in terms of the β variable:

$$f(\beta) = \exp\left(\frac{1}{2\pi} \frac{d\phi(\beta)}{d\beta}\right) \quad (44)$$

Given a frequency law, we can obtain the derivative of the Mellin phase and hence the expression of the signal in Mellin space $M^\xi[Z](\beta) = \exp(i\phi(\beta))$. This procedure is the analogous construction of a signal from time to frequency (and frequency to time) space using the definition of the instantaneous frequency (and the group delay). It only ensures that the signal will have, at one and the same time, a given bandwidth and spread in Mellin space but does not ensure the quality of the two autocorrelation functions in range and velocity spaces. For doing that, an interessant way is to extend the method of the stationary phase proposed in [8] on the relation (8) which allows to find a signal with given autocorrelation functions.

4. CONCLUSION

The analytical expression of the Cramer Rao bounds for velocity estimation in the broad-band case has been established using the Mellin transform. The most impressive result concerns the velocity resolution of active radar (and particularly sonar) which is not related to the inverse of the signal duration as in narrow-band case but to the inverse of the spread of the signal in Mellin space. This spread has a direct interpretation in the time-frequency half plane and can be easily estimated when duration, bandwidth and relative bandwidth are known. An interesting procedure to construct a signal with a given frequency law as a function of the Mellin variable has been proposed.

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References

[1] **H.L.Van Trees**, *"Detection, Estimation and Modulation Theory"*, Part I, II and III, John Wiley and Sons, New York 1971

[2] **P.M.Woodward**, *"Probability and Information Theory with Applications to Radar"*, Pergamon Press, New York 1953

[3] **J.Bertrand P.Bertrand and J.P.Ovarlez**, *"Discrete Mellin Transform for Signal Analysis"*, Proc. IEEE-ICASSP, Albuquerque, NM, USA 1990

[4] **J.P.Ovarlez**, *"La Transformation de Mellin: un Outil pour l'Analyse des Signaux à Large-Bande"*, Thesis University Paris 6, Paris April 1992.

[5] **J.P.Ovarlez J.Bertrand and P.Bertrand**, *"Computation of Affine Time-Frequency Distributions using the Fast Mellin Transform"*, Proc. IEEE-ICASSP, San Francisco, CA, USA 1992

[6] **J.Bertrand P.Bertrand and J.P.Ovarlez**, *"The Wavelet Approach in Radar Imaging and its Physical Interpretation"*, Proc. Int. Conf. Wavelets and Applications, Toulouse, France June 1992

[7] **J.Bertrand P.Bertrand and J.P.Ovarlez**, *"Compression d'Impulsions en Large Bande"*, Proc. XII Col. GRETSI, pp.21-24, Juan Les Pins, France June 1989

[8] **A.Papoulis**, *"Signal Analysis"*, McGraw Hill, New York 1977