First- and Second-Order Moments of the Normalized Sample Covariance Matrix of Spherically Invariant Random Vectors

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***Abstract*—Under Gaussian assumptions, the sample covariance matrix (SCM) is encountered in many covariance based processing algorithms. In case of impulsive noise, this estimate is no more appropriate. This is the reason why when the noise is modeled by spherically invariant random vectors (SIRV), a natural extension of the SCM is extensively used in the literature: the well-known normalized sample covariance matrix (NSCM), which estimates the covariance of SIRV. Indeed, this estimate gets rid of a fluctuating noise power and is widely used in radar applications. The aim of this paper is to derive closed-form expressions of the first- and second-order moments of the NSCM.**

***Index Terms*—Estimation, normalized sample covariance matrix (NSCM), performance analysis, spherically invariant random vectors (SIRV).**

I. Introduction

GIVEN independent identically distributed observations of a zero-mean complex Gaussian random vector, the sample covariance matrix (SCM) is the maximum likelihood estimate of the data covariance matrix. It is well known that the SCM is complex Wishart distributed, unbiased, and its second-order moments have simple expressions. The full statistical characterization of the SCM allows performance analysis of numerous algorithms relying on this estimate. However, this widespread estimate is no more appropriate when observations are not Gaussian. This is, for instance, the case for radar clutter returns [1], [2], radio fading analysis [3], or sonar interferences [4]. In these contexts, spherically invariant random vectors (SIRVs) have been appropriately used in modeling non-Gaussian problems. A SIRV is a complex compound Gaussian process with random power. More precisely, a SIRV $\mathbf{c}$ [5] is the product of the square root of a positive random variable τ, called the *texture*, and a m-dimensional independent zero-mean complex Gaussian vector $\mathbf{x}$ with covariance matrix $\mathbf{\Sigma}$ normalized according to $\mathrm{tr}(\mathbf{\Sigma}) = m$, where $\mathrm{tr}(.)$ is the trace of a matrix

$$\mathbf{c} = \sqrt{\tau}\mathbf{x}.$$

The notation $\mathbf{x} \sim \mathbb{CN}(\mathbf{0}, \mathbf{\Sigma})$ means that $\mathbf{x}$ is a zero mean complex Gaussian vector with covariance matrix $\mathbf{\Sigma}$. In this paper, we consider the estimation scheme of $\mathbf{\Sigma}$ from N independent SIRV observations, $\mathbf{c}_k = \sqrt{\tau_k}\mathbf{x}_k$, for $k = 1, \ldots, N$. In this context, we analyze the statistical properties of the well-known normalized sample covariance matrix (NSCM), introduced in [6], defined by

$$\widehat{\mathbf{\Sigma}} = \frac{m}{N}\sum_{k=1}^{N}\frac{\mathbf{c}_k\mathbf{c}_k^H}{\mathbf{c}_k^H\mathbf{c}_k} = \frac{m}{N}\sum_{k=1}^{N}\frac{\mathbf{c}_k\mathbf{c}_k^H}{\|\mathbf{c}_k\|^2} = \frac{m}{N}\sum_{k=1}^{N}\frac{\mathbf{x}_k\mathbf{x}_k^H}{\|\mathbf{x}_k\|^2} \quad (1)$$

where ^{H} denotes the transpose conjugate operator. Notice that the NSCM does not depend on the texture. The central limit theorem ensures that the NSCM is asymptotically Gaussian but first and second order moments of this estimate never appeared in the literature. Thus the goal of this paper is to fill these gaps when the $\mathbf{\Sigma}$-eigenvalues are distinct, i.e., the most common and realistic case.

II. First- and Second-Order Moments of the NSCM

In this section, we present the main results while computational details are provided in the Appendix.

Let us introduce the eigenvalue decomposition of $\mathbf{\Sigma}$

$$\mathbf{\Sigma} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H = \sum_{k=1}^{m}\lambda_k\mathbf{u}_k\mathbf{u}_k^H \quad (2)$$

where $\mathbf{\Lambda}$ is the diagonal matrix of the $\mathbf{\Sigma}$-eigenvalues, $\lambda_1 > \ldots > \lambda_m > 0$, and $\mathbf{U}$ is the unitary matrix of the $\mathbf{\Sigma}$-eigenvectors. Notice that we assume that all eigenvalues $\lambda_1, \ldots, \lambda_m$, are strictly positive and different, i.e. their multiplicity order is 1. We note $\mathbb{E}[.]$ the statistical mean.

Theorem: The first-order moment of the NSCM is given by

$$\mathbb{E}[\widehat{\mathbf{\Sigma}}] = m\mathbf{U}\mathbf{\Delta}\mathbf{U}^H \quad (3)$$

where

$$\delta_k = \sum_{\substack{n=1\\n\neq k}}^{m} d_n\lambda_k\left(\frac{\log\lambda_n - \log\lambda_k}{\lambda_n - \lambda_k} - \frac{1}{\lambda_n}\right) \quad (4)$$

$$d_n = \prod_{\substack{p=1\\p\neq n}}^{m}(1 - \lambda_p/\lambda_n)^{-1} \quad (5)$$

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and where $\boldsymbol{\Delta}$ is the diagonal matrix of the δ_k's, with $\delta_1 > \ldots > \delta_m > 0$.

Proof: See Appendix I. ■

Remark 1: This theorem provides as a by-product the eigen-decomposition of $\mathbb{E}[\widehat{\boldsymbol{\Sigma}}]$. It shows also that $\mathbb{E}[\widehat{\boldsymbol{\Sigma}}]$ and $\boldsymbol{\Sigma}$ share the same eigenvectors but have different eigenvalues. Consequently, the NSCM is a biased estimate of $\boldsymbol{\Sigma}$.

Remark 2: The NSCM preserves the ordering of the eigenvectors.

Let us denote $\mathrm{vec}(.)$ the operator which reshapes a $m \times n$ matrix elements into a mn column vector. Let us note $\mathbf{v} = \mathrm{vec}(\widehat{\boldsymbol{\Sigma}})$ and introduce the two matrices

$$\mathbf{V}_1 = \mathbb{E}[\mathbf{v}\mathbf{v}^H] \text{ and } \mathbf{V}_2 = \mathbb{E}[\mathbf{v}\mathbf{v}^T] \tag{6}$$

from which the covariances of the real and imaginary parts of the NSCM are straightforwardly derived.

Theorem 2: The NSCM is asymptotically Gaussian and

$$\begin{aligned}\mathbf{V}_1 = &\frac{m^2}{N}\sum_{p=1}^{m}\sum_{k=1}^{m}(w_{pk}+(N-1)\delta_p\delta_k)\\ &\times \mathrm{vec}\left(\mathbf{u}_p\mathbf{u}_p^H\right)\mathrm{vec}\left(\mathbf{u}_k\mathbf{u}_k^H\right)^H\\ &+\frac{m^2}{N}\sum_{p=1}^{m}\sum_{\substack{k=1\\k\neq p}}^{m} w_{pk}\mathrm{vec}\left(\mathbf{u}_p\mathbf{u}_k^H\right)\mathrm{vec}\left(\mathbf{u}_p\mathbf{u}_k^H\right)^H\end{aligned} \tag{7}$$

$$\begin{aligned}\mathbf{V}_2 = &\frac{m^2}{N}\sum_{p=1}^{m}\sum_{k=1}^{m}(w_{pk}+(N-1)\delta_p\delta_k)\\ &\times \mathrm{vec}\left(\mathbf{u}_p\mathbf{u}_p^H\right)\mathrm{vec}\left(\mathbf{u}_k\mathbf{u}_k^H\right)^T\\ &+\frac{m^2}{N}\sum_{p=1}^{m}\sum_{\substack{k=1\\k\neq p}}^{m} w_{pk}\mathrm{vec}\left(\mathbf{u}_p\mathbf{u}_k^H\right)\mathrm{vec}\left(\mathbf{u}_k\mathbf{u}_p^H\right)^T\end{aligned} \tag{8}$$

where

$$w_{kk} = \sum_{\substack{n=1\\n\neq k}}^{m} d_n\lambda_k\left(\frac{2\lambda_k\log(\lambda_k/\lambda_n)}{(\lambda_k-\lambda_n)^2} - \frac{1}{\lambda_n}\frac{\lambda_k+\lambda_n}{\lambda_k-\lambda_n}\right) \tag{9}$$

$$w_{pk} = \sum_{\substack{n=1\\n\neq p\\n\neq k}}^{m} d_n\tilde{w}_{pkn},\ for\ p\neq k \tag{10}$$

with

$$\begin{aligned}\tilde{w}_{pkn} = \lambda_p\lambda_k &\left\{\frac{\lambda_n(\lambda_p+\lambda_k)-2\lambda_p\lambda_k}{\lambda_n^2(\lambda_p-\lambda_k)^2} - \frac{\log(\lambda_n/\lambda_p)}{(\lambda_n-\lambda_p)(\lambda_n-\lambda_k)}\right.\\ &\left.+\left[\frac{\lambda_k(\lambda_n-\lambda_p)\log(\lambda_k/\lambda_p)}{\lambda_n^2(\lambda_n-\lambda_k)(\lambda_p-\lambda_k)^3}\right]\left(2\lambda_p\lambda_n-\lambda_p\lambda_k-\lambda_k^2\right)\right\}.\end{aligned} \tag{11}$$

where δ_n and d_n are, respectively, defined in (4) and (5).

Proof: See Appendix II. ■

III. Conclusion

The closed-form expressions of the first- and second-order moments of the NSCM for SIRV modeling have been provided in this paper with full detailed proofs. These analytical equations are essential for analyzing performance of signal-processing methods based on NSCM: detection schemes in radar applications and direction of arrivals estimation in array processing.

Appendix I
Proof of Theorem 1

Using the eigen-decomposition of (2), let us whiten $\mathbf{x}$ according to $\mathbf{y} = \boldsymbol{\Lambda}^{-1/2}\mathbf{U}^H\mathbf{x}$. Hence, $\mathbf{y} \sim \mathbb{CN}(\mathbf{0}, \mathbf{I})$ and

$$\frac{\mathbf{x}\mathbf{x}^H}{\|\mathbf{x}\|^2} = \mathbf{U}\boldsymbol{\Lambda}^{1/2}\frac{\mathbf{y}\mathbf{y}^H}{\mathbf{y}^H\boldsymbol{\Lambda}\mathbf{y}}\boldsymbol{\Lambda}^{1/2}\mathbf{U}^H.$$

The NSCM (1) statistical mean can be rewritten as

$$\mathbb{E}[\widehat{\boldsymbol{\Sigma}}] = m\mathbf{U}\boldsymbol{\Lambda}^{1/2}\mathbb{E}\left[\frac{\mathbf{y}\mathbf{y}^H}{\mathbf{y}^H\boldsymbol{\Lambda}\mathbf{y}}\right]\boldsymbol{\Lambda}^{1/2}\mathbf{U}^H. \tag{I.12}$$

Each component y_k of $\mathbf{y}$ is a zero-mean unit variance circular complex Gaussian variable and can be expressed as

$$y_k = \sqrt{\frac{1}{2}\chi_k^2}\exp(i\theta_k),$$

where χ_k^2 is Chi-squared-distributed with two degrees of freedom, θ_k is uniformly distributed on $[0, 2\pi]$. All the χ_k^2's and θ_k's are two-by-two independent. It follows that (I.12) yields

$$\mathbb{E}[\widehat{\boldsymbol{\Sigma}}] = m\sum_{k=1}^{m}\lambda_k\mathbb{E}\left[\chi_k^2\Big/\sum_{n=1}^{m}\lambda_n\chi_n^2\right]\mathbf{u}_k\mathbf{u}_k^H.$$

Let us set

$$\delta_k = \mathbb{E}\left[\lambda_k\chi_k^2\Big/\sum_{n=1}^{m}\lambda_n\chi_n^2\right] = \mathbb{E}\left[\frac{1}{1+X_2/X_1}\right] \tag{I.13}$$

where $X_1 = \lambda_k\chi_k^2$ and $X_2 = \sum_{\substack{n=1\\n\neq k}}^{m}\lambda_n\chi_n^2$.

The pdf of X_2 has to be derived to complete the proof. Since all χ_k^2's are independent, the characteristic function of X_2 is

$$\phi_{X_2}(u) = \prod_{\substack{n=1\\n\neq k}}^{m}(1-2i\lambda_n u)^{-1} = \sum_{\substack{n=1\\n\neq k}}^{m}\frac{c_n}{1-2i\lambda_n u}$$

where $c_n = \prod_{\substack{p=1\\p\neq n\\p\neq k}}^{m}(1-(\lambda_p/\lambda_n))^{-1}$. Thus, the pdf of X_2 follows

$$p_{X_2}(x) = \frac{1}{2}\sum_{\substack{n=1\\n\neq k}}^{m}\frac{c_n}{\lambda_n}\exp\left(-\frac{x}{2\lambda_n}\right), \quad x\geq 0. \tag{I.14}$$

So, the density of X_2 is obtained by the weighted sum of the densities of $\lambda_n\chi_n^2$ by the coefficient c_n. Now, the pdf of the ratio X_2/X_1 is a weighted sum of Fisher-Snedecor (F) laws

$$p_{X_2/X_1}(x) = \sum_{\substack{n=1\\n\neq k}}^{m}c_n\frac{\lambda_k}{\lambda_n}\left(1+\frac{\lambda_k}{\lambda_n}x\right)^{-2}, \quad x\geq 0 \tag{I.15}$$

and after some manipulations, (I.13) yields

$$\delta_k = \sum_{\substack{n=1\\n\neq k}}^{m}c_n\left(\frac{\lambda_n/\lambda_k}{(1-\lambda_n/\lambda_k)^2}\log(\lambda_n/\lambda_k)+\frac{1}{1-\lambda_n/\lambda_k}\right).$$

It remains to show that $\delta_1 > \ldots > \delta_m > 0$. First, the δ_k's, defined in (I.13), are strictly positive. Now, let us consider the following function for $x > 0$ and $y > 0$:

$$f_w(x,y) = \frac{1}{4}\int_{\mathbb{R}_+^2}\frac{xu}{xu+yv+w}\exp\left(-\frac{u+v}{2}\right)dudv.$$

It follows from (I.13) that we have $\delta_k = \mathbb{E}_w[f_w(\lambda_k,\lambda_p)]$ and $\delta_p = \mathbb{E}_w[f_w(\lambda_p,\lambda_k)]$, for $w = \sum_{\substack{n=1\\ n\neq k\\ n\neq p}}^m \lambda_n\chi_n^2$, and where $\mathbb{E}_w[.]$ stands for the statistical mean related to w. To show that $\delta_k < \delta_p$, we prove that $f_w(\lambda_k,\lambda_p) < f_w(\lambda_p,\lambda_k)$ for all w, assuming $\lambda_k < \lambda_p$. Let us define the functions

$$f_1(t) = f_w\left((1-t)\lambda_p + t\lambda_k, \lambda_p\right)$$
$$f_2(t) = f_w\left(\lambda_p, (1-t)\lambda_p + t\lambda_k\right)$$

which verify $f_1(0) = f_2(0)$, $f_1(1) = \delta_k$, and $f_2(1) = \delta_p$. To demonstrate that $\delta_k < \delta_p$, we show hereafter that f_1 and f_2 are, respectively, strictly decreasing and strictly increasing functions of t on the interval [0,1]. We have

$$\frac{\partial f_w}{\partial x}(x,y) = \frac{1}{4}\int_{\mathbb{R}_+^2} \frac{u(yv+w)}{(xu+yv+w)^2} e^{-(u+v)/2} du dv > 0$$
$$\frac{\partial f_w}{\partial y}(x,y) = -\frac{1}{4}\int_{\mathbb{R}_+^2} \frac{vxu}{(xu+yv+w)^2} e^{-(u+v)/2} du dv < 0$$

from which we obtain

$$\frac{df_1}{dt} = \left(\frac{\partial f_w}{\partial x}\right)_{((1-t)\lambda_p + t\lambda_k, \lambda_p)} (\lambda_k - \lambda_p) < 0$$
$$\frac{df_2}{dt} = \left(\frac{\partial f_w}{\partial y}\right)_{(\lambda_p, (1-t)\lambda_p + t\lambda_k)} (\lambda_k - \lambda_p) > 0.$$

In summary, $\delta_k < \delta_p$ for any k, p such that $\lambda_k < \lambda_p$. This completes the proof of Theorem 1.

Appendix II
Proof of Theorem 2

By expressing the variance of the NSCM as a linear combination of functions of the $\mathbf{\Sigma}$-eigenvectors, we compute the statistical means of the coefficients. Equations (1), (3), (6), and (I.12) lead to

$$\mathbf{V}_1 = \frac{m^2}{N^2}\sum_{k=1}^m\sum_{p=1}^m \mathbb{E}\left[\mathrm{vec}\left(\frac{\mathbf{x}_k\mathbf{x}_k^H}{||\mathbf{x}_k||^2}\right)\mathrm{vec}\left(\frac{\mathbf{x}_p\mathbf{x}_p^H}{||\mathbf{x}_p||^2}\right)^H\right]$$
$$= \frac{m^2}{N}\sum_{p,j,n,k}^m \Big\{ [\omega_{pjnk} + (N-1)\delta_p\delta_n\delta(p-j)\delta(n-k)] \times \mathrm{vec}\left(\mathbf{u}_p\mathbf{u}_j^H\right)\mathrm{vec}\left(\mathbf{u}_n\mathbf{u}_k^H\right)^H \Big\}$$

where

$$\omega_{pjnk} = \mathbb{E}\left[\frac{\sqrt{\lambda_p\lambda_j\lambda_n\lambda_k\chi_p^2\chi_j^2\chi_n^2\chi_k^2}}{\left(\sum_{t=1}^m \lambda_t\chi_t^2\right)^2}\right]\mathbb{E}\left[e^{i(\theta_p-\theta_j+\theta_k-\theta_n)}\right]$$

and $\delta(.)$ is the Kronecker delta. The θ's being independent uniform variables, the last term of previous equations is zero unless $p=j$, $k=n$ or $p=n$, $k=j$, which leads to

$$\mathbf{V}_1 = \frac{m^2}{N}\sum_{p,k}^m [w_{pk} + (N-1)\delta_p\delta_k]\,\mathrm{vec}\left(\mathbf{u}_p\mathbf{u}_p^H\right)\mathrm{vec}\left(\mathbf{u}_k\mathbf{u}_k^H\right)^H + \frac{m^2}{N}\sum_{p=1}^m\sum_{\substack{k=1\\ k\neq p}}^m w_{pk}\mathrm{vec}\left(\mathbf{u}_p\mathbf{u}_k^H\right)\mathrm{vec}\left(\mathbf{u}_p\mathbf{u}_k^H\right)^H$$

where

$$w_{pk} = \lambda_p\lambda_k\mathbb{E}\left[\chi_p^2\chi_k^2 / \left(\sum_{n=1}^m \lambda_n\chi_n^2\right)^2\right]. \qquad \text{(II.16)}$$

This is (7) of Theorem 2 and (8) is derived from the same reasoning. Concerning (9), one has, from (II.16), for $p=k$, $w_{kk} = \mathbb{E}[(1+X_2/X_1)^{-2}]$, where X_1 and X_2 are defined in (I.13). Thus $w_{kk} = \int_0^{+\infty}(1+x)^{-2}p_{X_2/X_1}(x)dx$. Equation (10) is derived further. The proof needs some results related to exponential integrals introduced hereafter.

Exponential Integrals and Related Functions

From [7, p. 228], let us recall the definition of the exponential integral

$$E_{\mathrm{n}}(z) = \int_1^{+\infty} \frac{e^{-zt}}{t^n}dt,\ n\in\mathbb{N},\ \Re(z) > 0$$
$$E_1(z) = -\gamma - \ln z - \sum_{n=1}^{+\infty}\frac{(-1)^n z^n}{n n!} \qquad \text{(III.17)}$$

where $\Re(z)$ denotes the real part of z and γ is Euler's gamma constant. Let us introduce the real function

$$F_n(a,x) = \int_x^{+\infty} t^n e^{-at}E_1(t)dt,\ n\in\mathbb{N},\ x>0, a>-1. \qquad \text{(III.18)}$$

Let us show that the integral involved in the definition of $F_n(a,x)$ is well defined for $x>0$ and $a>-1$. From [7, p. 229], we have $e^x E_1(x) < \log(1+1/x)$ for $x>0$, which leads to

$$0 \le F_n(a,x) \le \log(1+1/x)x^{n+1}\alpha_n\left([a+1]x\right)$$

where the function $\alpha_n(y) = \int_1^{+\infty} t^n e^{-yt}dt$, $n\in\mathbb{N}$, is defined for $y>0$, see [7, p. 228]. In conclusion, function $F_n(a,x)$ is well defined for $x>0$ and $(a+1)x>0$, i.e., for $x>0$ and $a>-1$.

We are interested in the limiting values of $F_n(a,x)$ when x tends to zero. Integration by parts leads to

$$F_n(a,x) = \frac{x^n}{a}e^{-ax}E_1(x) + \frac{n}{a}F_{n-1}(a,x) - \frac{x^n}{a}\alpha_{n-1}\left([a+1]x\right) \qquad \text{(III.19)}$$

where $\alpha_n(y)$ is given by [7, p. 228]

$$\alpha_n(y) = n!y^{-n-1}e^{-y}\left(1 + y + \frac{y^2}{2!} + \ldots + \frac{y^n}{n!}\right).$$

Equation (III.19) combined with $\lim_{x\to 0} x^n E_1(x) = 0$ for $n \ge 1$ which results from the series expansion (III.17) with $\lim_{x\to 0} F_0(a,x) = \int_{\mathbb{R}_+} e^{-at}E_1(t)dt = \ln(1+a)/a$, see [7, p. 230], and with the above expression of $\alpha_n(y)$, leads to

$$\lim_{x\to 0} F_1(a,x) = F_1(a,0) = \frac{\ln(1+a)}{a^2} - \frac{1}{a(1+a)}$$
$$\lim_{x\to 0} F_2(a,x) = F_2(a,0) = \frac{2\ln(1+a)}{a^3} - \frac{3a+2}{a^2(1+a)^2}. \qquad \text{(III.20)}$$

END OF PROOF OF THEOREM 2 (w_{pk} FOR $p \neq k$, SEE (10))

It remains to compute (II.16) to complete the proof of Theorem 2. Let us write $w_{pk} = \mathbb{E}[(1 + \tilde{X}_2/\tilde{X}_1)^{-1}]$ with $\tilde{X}_1 = \lambda_k \chi_k^2 + \lambda_p \chi_p^2$ and $\tilde{X}_2 = \sum_{\substack{n=1 \\ n\neq k \\ n \neq p}}^m \lambda_n \chi_n^2$. A pdf decomposition similar to (I.14), but for $\tilde{X}_2$, provides

$$w_{pk} = \sum_{\substack{n=1 \\ n\neq p \\ n\neq k}}^m \delta_{pkn} \prod_{\substack{j=1 \\ j\neq n \\ j \neq p \\ j\neq k}}^m \left(1 - \frac{\lambda_j}{\lambda_n}\right)^{-1} \tag{IV.21}$$

where $\delta_{pkn} = \lambda_p \lambda_k \mathbb{E}[\chi_p^2 \chi_k^2/(\lambda_p \chi_p^2 + \lambda_k \chi_k^2 + \lambda_n \chi_n^2)^2]$ is

$$\delta_{pkn} = \frac{\lambda_p \lambda_k}{8} \int_{\mathbb{R}_+^3} \frac{x_p x_k e^{-(x_p + x_k + x_n)/2}}{(\lambda_p x_p + \lambda_k x_k + \lambda_n x_n)^2} dx_p dx_k dx_n.$$

An analytic expression of δ_{pkn} is obtained by computing the above integral. The previous equation is rewritten as

$$\delta_{pkn} = \frac{\lambda_p \lambda_k}{8} \int_{\mathbb{R}_+^2} t_1 x_p x_k e^{-(x_p + x_k)/2} dx_p dx_k \tag{IV.22}$$

where $t_1 = \int_0^{+\infty} e^{-x_n/2} (\lambda_p x_p + \lambda_k x_k + \lambda_n x_n)^{-2} dx_n$. Then, by setting $C = \lambda_p x_p + \lambda_k x_k$, t_1 is rewritten as

$$\begin{aligned} t_1 &= \frac{1}{\lambda_n C} \exp\left(\frac{C}{2\lambda_n}\right) E_2\left(\frac{C}{2\lambda_n}\right) \\ &= \frac{1}{\lambda_n C}\left[1 - \frac{C}{2\lambda_n} \exp\left(\frac{C}{2\lambda_n}\right) E_1\left(\frac{C}{2\lambda_n}\right)\right]. \end{aligned}$$

where E_1 and E_2 are defined in (III.17). Now, by replacing t_1 in (IV.22), we obtain

$$\begin{aligned} \delta_{pkn} &= \frac{\lambda_p \lambda_k}{8\lambda_n}\left(t_2 - \frac{1}{2\lambda_n} t_3\right) \\ t_2 &= \int_{\mathbb{R}_+^2} \frac{x_p x_k}{\lambda_p x_p + \lambda_k x_k} e^{-(x_p + x_k)/2} dx_p dx_k \\ t_3 &= \int_{\mathbb{R}_+^2} x_p x_k \exp\left(-\frac{1}{2}\left(x_p + x_k - \frac{\lambda_p x_p + \lambda_k x_k}{\lambda_n}\right)\right) \\ &\quad \times E_1\left(\frac{\lambda_p x_p + \lambda_k x_k}{2\lambda_n}\right) dx_p dx_k. \end{aligned} \tag{IV.23}$$

Integrating firstly along x_k in t_2 allows to rewrite t_2 as

$$t_2 = \frac{8}{\lambda_k} - \frac{8\lambda_k}{\lambda_p^2} F_2\left(\frac{\lambda_k - \lambda_p}{\lambda_p}, 0\right) \tag{IV.24}$$

where the function $F_2(.)$ is defined in (III.20). Now, let us compute t_3 as

$$t_3 = \int_0^{+\infty} x_p \exp\left(-\frac{x_p}{2}\left(1 - \frac{\lambda_p}{\lambda_n}\right)\right) t_4 dx_p$$

with $t_4 = \int_{\mathbb{R}_+} x_k e^{-x_k(1-\lambda_k/\lambda_n)/2} E_1((\lambda_p x_p + \lambda_k x_k)/2\lambda_n) dx_k$. By a change of variable, t_4 is rewritten as

$$\begin{aligned} t_4 &= \frac{2\lambda_n}{\lambda_k^2} \int_{\frac{\lambda_p x_p}{2\lambda_n}}^{+\infty} (2\lambda_n t - \lambda_p x_p) e^{-\frac{2\lambda_n t - \lambda_p x_p}{2\lambda_k}\left(1 - \frac{\lambda_k}{\lambda_n}\right)} E_1(t) dt \\ &= \frac{2\lambda_n}{\lambda_k^2} e^{\lambda_p xp/2/\lambda_k(1-\lambda_k/\lambda_n)} \\ &\quad \times \left[2\lambda_n F_1\left(\frac{\lambda_n}{\lambda_k} - 1, \frac{\lambda_p x_p}{2\lambda_n}\right) - \lambda_p x_p F_0\left(\frac{\lambda_n}{\lambda_k} - 1, \frac{\lambda_p x_p}{2\lambda_n}\right)\right]. \end{aligned}$$

and can be simplified with (III.19) and with $F_0(a, x) = e^{-ax} E_1(x)/a - E_1([a+1]x)/a$. The simplified expression of t_4 allows to rewrite t_3 as

$$\begin{aligned} t_3 = \frac{16\lambda_n^2}{\lambda_n - \lambda_k} \Bigg(&\frac{1}{\lambda_p^2(\lambda_n - \lambda_k)}\left[\lambda_n^2 F_1(b_n, 0) - \lambda_k^2 F_1(b_k, 0)\right] \\ &- \frac{1}{\lambda_n} + \frac{\lambda_k^2}{\lambda_n \lambda_p^2} F_2(b_k, 0)\Bigg) \end{aligned}$$

where $b_j = (\lambda_j - \lambda_p)/\lambda_p$, for $j = k, n$. Finally, combining the previous result with (III.20), (IV.23) and (IV.24), one has

$$\begin{aligned} \delta_{pkn} = \lambda_p \lambda_k \Bigg[&\frac{\lambda_n(\lambda_p + \lambda_k) - 2\lambda_p \lambda_k}{(\lambda_n - \lambda_p)(\lambda_n - \lambda_k)(\lambda_p - \lambda_k)^2} \\ &- \frac{\lambda_n^2 \log \lambda_n}{(\lambda_n - \lambda_p)^2 (\lambda_n - \lambda_k)^2} + \frac{\lambda_p \lambda_k \log \lambda_k}{(\lambda_p - \lambda_k)^3} \\ &\times \frac{2\lambda_p \lambda_n - \lambda_p \lambda_k - \lambda_k^2}{\lambda_p(\lambda_n - \lambda_k)^2} - \frac{\lambda_p \lambda_k}{(\lambda_p - \lambda_k)^3} \\ &\times \frac{2\lambda_k \lambda_n - \lambda_p \lambda_k - \lambda_p^2}{\lambda_k(\lambda_n - \lambda_p)^2} \log \lambda_p\Bigg]. \end{aligned}$$

Thanks to (IV.21), the previous equation provides (10) and (11). This concludes the proof of Theorem 2.

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