ON PERSYMMETRIC COVARIANCE MATRICES IN ADAPTIVE DETECTION

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ABSTRACT

In the general area of radar detection, estimation of the clutter covariance matrix is an important point. This matrix commonly exhibits a persymmetric structure: this is the case for instance for active systems using a symmetrically spaced linear array or pulse train. In this context, this paper provides a new Gaussian adaptive detector called the Persymmetric Adaptive Matched Filter (P-AMF). Its theoretical distribution is derived allowing adjustment of the detection threshold for a given Probability of False Alarm (PFA). Simulations results highlight the improvement in term of probability of detection (PD) of the P-AMF in comparison with the classical Adaptive Matched Filter (AMF).

Index Terms— Adaptive signal detection, Parameter estimation, Maximum Likelihood Estimation, Covariance matrices, Radar detection.

1. INTRODUCTION

One of the major problems in radar detection consists in detecting a known signal $\mathbf{p} \in \mathbb{C}^m$ corrupted by an additive clutter $\mathbf{c}$. Classically, this problem can be stated as the following binary hypothesis test:

$$\begin{cases} H_0 : \mathbf{y} = \mathbf{c}, & \mathbf{y}_k = \mathbf{c}_k, \text{ for } 1 \le k \le K\,, \\ H_1 : \mathbf{y} = A\,\mathbf{p} + \mathbf{c}, & \mathbf{y}_k = \mathbf{c}_k, \text{ for } 1 \le k \le K\,, \end{cases} \quad (1)$$

where $\mathbf{y}$ is the complex m-vector of the received signal, A is an unknown complex target amplitude and $\mathbf{c}$ is a complex zero-mean Gaussian m-vector with covariance matrix $\mathbf{M} = E[\mathbf{c}\,\mathbf{c}^H]$. Under both hypotheses, it is assumed that K signal-free data $\mathbf{y}_k$ are available for clutter parameters estimation. The $\mathbf{y}_k$'s are the so-called secondary data. They are independent and identically distributed (i.i.d) with the same distribution as $\mathbf{c}$.

In the sequel, the real (resp. complex) Gaussian distribution with zero-mean and covariance matrix $\mathbf{M}$ is denoted by $\mathcal{N}(\mathbf{0}, \mathbf{M})$ (resp. $\mathcal{CN}(\mathbf{0}, \mathbf{M})$), $E[.]$ stands for the expectation operator, ^{H} denotes the conjugate transpose, * the conjugate and $^\top$ the transpose operator, $\|\ .\ \|$ is the usual $\mathcal{L}^2$-norm, $\mathbf{I}_m$ is the *m*-th order identity matrix and $\sim$ means "distributed as".

When $\mathbf{M}$ is known, the Generalized Likelihood Ratio Test (GLRT) is referred to as the Optimum Gaussian Detector (OGD):

$$\Lambda_{OGD} = \frac{|\mathbf{p}^H\,\mathbf{M}^{-1}\,\mathbf{y}|^2}{\mathbf{p}^H\,\mathbf{M}^{-1}\,\mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda\,, \quad (2)$$

where the detection threshold λ is related to the PFA by: $\lambda = \sqrt{-\ln(P_{fa})}$. However, the clutter covariance matrix $\mathbf{M}$ is generally unknown and has to be estimated. For that purpose, the Maximum Likelihood theory provides the well-known Sample Covariance Matrix (SCM) built from the secondary data and defined by:

$$\widehat{\mathbf{M}}_{SCM} = \frac{1}{K}\sum_{k=1}^{K} \mathbf{y}_k\,\mathbf{y}_k^H\,. \quad (3)$$

Then, substituting $\widehat{\mathbf{M}}_{SCM}$ for $\mathbf{M}$ in (2) leads to the so-called Adaptive Matched Filter (AMF) test [1]:

$$\Lambda_{AMF} = \frac{|\mathbf{p}^H\,\widehat{\mathbf{M}}_{SCM}^{-1}\,\mathbf{y}|^2}{\mathbf{p}^H\,\widehat{\mathbf{M}}_{SCM}^{-1}\,\mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda\,. \quad (4)$$

The AMF has the Constant False Alarm Rate (CFAR) property and its distribution is known (see e.g. [1, 2, 3]). However, the AMF exhibits a detection loss in comparison with the OGD. Moreover, $\widehat{\mathbf{M}}_{SCM}$ defined by (3) does not take into account any prior knowledge on the clutter covariance structure.

Many applications can result in a clutter covariance matrix that exhibits some particular structure. Such a situation is frequently met in radar systems using a symmetrically spaced linear array for spatial domain processing, or symmetrically spaced pulse train for temporal domain processing [4, 5]. In these systems, the clutter covariance matrix $\mathbf{M}$ has the persymmetric property:

$$\mathbf{M} = \mathbf{J}_m\,\mathbf{M}^*\,\mathbf{J}_m\,, \quad (5)$$

where $\mathbf{J}_m$ is the m-dimensional antidiagonal matrix having 1 as non-zero elements. The signal vector is also persymmetric, i.e. it satisfies:

$$\mathbf{p} = \mathbf{J}_m\,\mathbf{p}^*\,. \quad (6)$$

The persymmetric structure of $\mathbf{M}$ can be exploited to improve its estimation accuracy compared to the SCM. This approach

has been followed by [4] in the framework of Kelly's GLRT [6]. Our paper is based on another widespread detection test, the AMF. It studies its statistical performance when using in (4) the persymmetric Maximum Likelihood (ML) estimate of the clutter covariance matrix instead of the SCM. The resulting detection test is called P-AMF for Persymmetric Adaptive Matched Filter. One of the main contribution is the derivation of the P-AMF distribution under hypothesis H_0. This allows the theoretical setting of the detection threshold for a given PFA.
This paper is organized as follows. Section 2 shows how the persymmetric structure of the covariance matrix can be exploited to provide the new P-AMF. Section 3 derives the statistical distribution of the P-AMF under hypothesis H_0. Section 4 presents simulation results which illustrate the improvement in terms of detection performance of the P-AMF on the conventional AMF.

2. PROBLEM STATEMENT AND PRELIMINARIES

In the context of persymmetric $\mathbf{M}$ and $\mathbf{p}$, the problem defined by (1) will be first reformulated thanks to the following proposition.

Proposition 2.1
Let $\boldsymbol{T}$ be the unitary matrix defined as:

$$\boldsymbol{T} = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} \boldsymbol{I}_{m/2} & \boldsymbol{J}_{m/2} \\ i\boldsymbol{I}_{m/2} & -i\boldsymbol{J}_{m/2} \end{pmatrix} & \text{for } m \text{ even} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} \boldsymbol{I}_{(m-1)/2} & 0 & \boldsymbol{J}_{(m-1)/2} \\ 0 & \sqrt{2} & 0 \\ i\boldsymbol{I}_{(m-1)/2} & 0 & -i\boldsymbol{J}_{(m-1)/2} \end{pmatrix} & \text{for } m \text{ odd.} \end{cases} \quad (7)$$

Persymmetric vectors and Hermitian matrices are characterized by the following properties:

- *$\boldsymbol{p} \in \mathbb{C}^m$ is a persymmetric vector if and only if $\boldsymbol{T}\,\boldsymbol{p}$ is a real vector.*
- *$\boldsymbol{M}$ is a persymmetric Hermitian matrix if and only if $\boldsymbol{T}\,\boldsymbol{M}\,\boldsymbol{T}^H$ is a real symmetric matrix.*

Proof 2.1
The proof is straightforward and involves elementary algebraic manipulations.

Using previous proposition, the original problem (1) can be equivalently reformulated as follows. Let us introduce the transformed primary data $\mathbf{x}$, the transformed secondary data $\mathbf{x}_k$, the transformed clutter vector $\mathbf{n}$, the transformed signal vector $\mathbf{s}$ and the transformed clutter covariance matrix $\mathbf{R}$ defined as: $\mathbf{x} = \mathbf{T}\,\mathbf{y}$, $\mathbf{x}_k = \mathbf{T}\,\mathbf{y}_k$, $\mathbf{s} = \mathbf{T}\,\mathbf{p}$, $\mathbf{n} = \mathbf{T}\,\mathbf{c}$, $\mathbf{n}_k = \mathbf{T}\,\mathbf{c}_k$ and $\mathbf{R} = E(\mathbf{n}\,\mathbf{n}^H) = E(\mathbf{n}_k\,\mathbf{n}_k^H) = \mathbf{T}\,\mathbf{M}\,\mathbf{T}^H$.

From proposition 2.1, the transformed signal vector $\mathbf{s}$ and the transformed clutter covariance matrix are both real. Then, the original problem (1) is equivalent to:

$$\begin{cases} H_0 : \mathbf{x} = \mathbf{n} & \mathbf{x}_k = \mathbf{n}_k, \text{ for } 1 \le k \le K\,, \\ H_1 : \mathbf{x} = A\,\mathbf{s} + \mathbf{n} & \mathbf{x}_k = \mathbf{n}_k, \text{ for } 1 \le k \le K\,, \end{cases} \quad (8)$$

where $\mathbf{x} \in \mathbb{C}^m$, $\mathbf{n} \sim \mathcal{CN}(0, \mathbf{R})$, $\mathbf{s}$ is a known real vector, $\mathbf{R}$ is an unknown real symmetric matrix. The K transformed secondary data $\mathbf{x}_k$ are i.i.d and share the same $\mathcal{CN}(0, \mathbf{R})$ distribution as $\mathbf{n}$. From now on, the problem under study is the problem defined by (8).
Let us now investigate the ML estimate of the real covariance matrix $\mathbf{R}$ from the K secondary data $\mathbf{x}_k$. The main motivation for introducing the transformed data is that the resulting distribution of the ML estimate of $\mathbf{R}$ is very simple. This was not the case in [7] when dealing with the original secondary data $\mathbf{y}_k$ with persymmetric covariance matrix.

Proposition 2.2
The ML estimate $\widehat{\boldsymbol{R}}$ of real matrix $\boldsymbol{R}$ is unbiased and is given by:

$$\widehat{\mathbf{R}} = \mathcal{R}e(\widehat{\mathbf{R}}_{SCM})\,, \quad (9)$$

where $\mathcal{R}e(.)$ stands for the real part and where:

$$\widehat{\boldsymbol{R}}_{SCM} = \frac{1}{K}\sum_{k=1}^{K} \boldsymbol{x}_k\,\boldsymbol{x}_k^H = \boldsymbol{T}\,\widehat{\mathbf{M}}_{SCM}\,\boldsymbol{T}^H\,. \quad (10)$$

$2\,K\,\widehat{\boldsymbol{R}}$ is real Wishart distributed with $2K$ degrees of freedom.

Proof 2.2
The proof is straightforward and is omitted.

Actually, taking into account the real structure of $\mathbf{R}$ (or equivalently the persymmetric structure of $\mathbf{M}$) in the ML estimation procedure allows to virtually double the number of secondary data. Let us now consider the AMF for the detection problem (8) based on the estimate $\widehat{\mathbf{R}}$ defined by (9). This leads to the following detection test, called the P-AMF,

$$\Lambda_{PAMF} = \frac{|\mathbf{s}^\top \widehat{\mathbf{R}}^{-1}\mathbf{x}|^2}{\mathbf{s}^\top \widehat{\mathbf{R}}^{-1}\mathbf{s}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda\,, \quad (11)$$

or equivalently, in terms of the original data,

$$\Lambda_{PAMF} = \frac{|\mathbf{p}^H\mathbf{T}^H[\mathcal{R}e(\mathbf{T}\widehat{\mathbf{M}}_{SCM}\mathbf{T}^H)]^{-1}\mathbf{T}\mathbf{y}|^2}{\mathbf{p}^H\mathbf{T}^H[\mathcal{R}e(\mathbf{T}\widehat{\mathbf{M}}_{SCM}\mathbf{T}^H)]^{-1}\mathbf{T}\mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda\,. \quad (12)$$

Note that $\mathbf{s}$ and $\widehat{\mathbf{R}}$ are real in (11), while $\mathbf{x}$ is complex. The aim of this paper is twofold:

- to derive the statistical distribution of Λ_{PAMF} under hypothesis H_0 in order to set the detection threshold λ for a given PFA;
- to emphasize the improvement of the P-AMF on the AMF in terms of probability of detection.

3. STATISTICAL ANALYSIS OF THE P-AMF

The purpose of this section is to derive the statistical distribution of the P-AMF and, as a consequence, to establish the relationship between the PFA and the detection threshold λ.

Proposition 3.1

• *Under H_0, the Probability Density Function (PDF) of Λ_{PAMF}, defined by (11), is:*

$$p(z) = \frac{(2K-m+1)\,(2K-m+2)}{2\,K\,(2K+1)} \times {}_2F_1\left(\frac{2K-m+3}{2}, \frac{2K-m+4}{2}; \frac{2K+3}{2}; -\frac{z}{K}\right), \tag{13}$$

where ${}_2F_1$ is the hypergeometric function [8] defined by

$${}_2F_1(a,b;c;\lambda) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}\,(1-t)^{c-b-1}}{(1-t\lambda)^a}\,dt\,.$$

• *The relationship between the PFA and the detection threshold λ is thus:*

$$PFA = {}_2F_1\left(\frac{2K-m+1}{2}, \frac{2K-m+2}{2}; \frac{2K+1}{2}; -\frac{\lambda}{K}\right).$$

Proof 3.1

Due to limited space, the proof is just outlined. Let $\mathbf{e}_1 = (1,0,...,0)^\top$, let $\mathbf{R} = \mathbf{R}^{1/2}\,\mathbf{R}^{\top/2}$ be a standard factorization of $\mathbf{R}$ and let $\boldsymbol{Q}$ be a real unitary matrix such that $\mathbf{e}_1 = \mathbf{Q}\,\mathbf{R}^{-1/2}\,\mathbf{s}$. Let us set

$$\widehat{\boldsymbol{W}} = 2K\,\mathbf{Q}\,\mathbf{R}^{-1/2}\,\widehat{\mathbf{R}}\,\mathbf{R}^{-\top/2}\,\mathbf{Q}^\top \text{ and } \mathbf{z} = \mathbf{Q}\mathbf{R}^{-1/2}\mathbf{x}\,.$$

$\widehat{\boldsymbol{W}}$ is real Wishart distributed with $2K$ degrees of freedom and parameter matrix $\mathbf{I}_m$, $\mathbf{z} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I}_m)$. Then, the test statistic Λ_{PAMF} (11) may be rewritten as:

$$\Lambda_{PAMF} = 2K \left(\frac{\boldsymbol{e}_1^\top\,\hat{\boldsymbol{W}}^{-2}\,\boldsymbol{e}_1}{\boldsymbol{e}_1^\top\,\hat{\boldsymbol{W}}^{-1}\,\boldsymbol{e}_1}\right) \frac{|\boldsymbol{e}_1^\top\,\hat{\boldsymbol{W}}^{-1}\,\boldsymbol{x}|^2}{\boldsymbol{e}_1^\top\,\hat{\boldsymbol{W}}^{-2}\,\boldsymbol{e}_1} = 2\,K\,b\,a\,. \tag{14}$$

The conditional distribution of $2a$ given $\widehat{\boldsymbol{W}}$ is a Chi square distribution with 2 degrees of freedom denoted by χ^2_2. This distribution does not involve $\widehat{\boldsymbol{W}}$, and $a \sim \frac{1}{2}\chi^2_2$ is therefore independent of b. Now, to derive the PDF of b, we use the Bartlett matrix decomposition [9] $\hat{\boldsymbol{W}} = \boldsymbol{U}.\boldsymbol{U}^\top$ where $\boldsymbol{U} = (u_{i,j})_{1\le i\le j\le m}$ is an upper triangular matrix whose random elements are independent and distributed as:

$$u_{i,i}^2 \sim \chi^2_{2K+i-m} \text{ and } u_{i,j} \sim \mathcal{N}(0,1) \text{ for } i<j.$$

Let $u'_{i,j}$ be the elements of the matrix $\boldsymbol{U}^{-1}$, $\boldsymbol{u}'_{1,k}$ the vector built with the $k \le m$ first non-zero elements of the first row of $\boldsymbol{U}^{-1}$ and $\boldsymbol{u}_{k,k}$ the vector built with the $k \le m$ first non-zero elements of the k-th column of $\boldsymbol{U}$. Given that $\boldsymbol{U}^{-1}\,\boldsymbol{U} = \boldsymbol{I}_m$, the upper-left corner element of $\boldsymbol{U}^{-1}$ is $u'_{1,1} = u^{-1}_{1,1}$ and the vector $\boldsymbol{u}'_{1,k}$ must satisfy $\boldsymbol{u}'^\top_{1,k}\,\boldsymbol{u}_{k,k} + u'_{1,k+1}\,u_{k+1,k+1} = 0$, for $1<k<m$. This leads to:

$$u'^2_{1,k+1} = \frac{|\boldsymbol{u}'^\top_{1,k}\,\boldsymbol{u}_{k,k}|^2}{\|\,\boldsymbol{u}'_{1,k}\,\|^2}\,\frac{\|\,\boldsymbol{u}'_{1,k}\,\|^2}{u^2_{k+1,k+1}} = \alpha_k\,\frac{\|\,\boldsymbol{u}'_{1,k}\,\|^2}{u^2_{k+1,k+1}}\,. \tag{15}$$

Conditioned to $\boldsymbol{u}'_{1,k}$, α_k is χ^2_1-distributed and is independent of $\boldsymbol{u}'_{1,k}$. By using the relation $\boldsymbol{U}^{-1}\,\mathbf{e}_1 = u'_{1,1}\,\mathbf{e}_1$, the random variable b can be written as:

$$b = \boldsymbol{e}_1^\top\,\boldsymbol{U}^{-1}\,\boldsymbol{U}^{-\top}\,\mathbf{e}_1 = \|\,\boldsymbol{e}_1^\top\,\boldsymbol{U}^{-1}\,\|^2 = \|\,\boldsymbol{u}'_{1,m}\,\|^2\,. \tag{16}$$

Now, notice that $\|\,\boldsymbol{u}'_{1,m}\,\|^2 = \|\,\boldsymbol{u}'_{1,m-1}\,\|^2 + u'^2_{1,m}$. From (15), one has:

$$b = \|\,\boldsymbol{u}'_{m-1}\,\|^2 \left(1 + \frac{\alpha_{m-1}}{u^2_{m,m}}\right) = \frac{1}{u^2_{1,1}} \prod_{k=2}^{m} \left(1 + \frac{\alpha_{k-1}}{u^2_{k,k}}\right),$$

where α_k are all independent, are independent of $u^2_{k,k}$ and are χ^2_1-distributed. Since $\left(1 + \frac{\alpha_{k-1}}{u^2_{k,k}}\right)^{-1} \sim \beta_1\left(\frac{2K-m+k}{2}, \frac{1}{2}\right)$ and

$$\prod_{k=2}^{m} \frac{1}{\beta_1\left(\frac{2K-m+k}{2}, \frac{1}{2}\right)} \sim \frac{1}{\beta_1\left(\frac{2K-m+2}{2}, \frac{m-1}{2}\right)},$$

where the β_1's are (first kind) Beta distributed random variables. This leads to the distribution of b and consequently to the PDF (13).

Moreover, $Pfa = \int_\lambda^{+\infty} p(z)\,dz$ which concludes the proof.

4. SIMULATIONS

This section presents some simulations with vector data of order $m = 20$. To illustrate previous results, we first plot on figure 1 the PFA versus the P-AMF detection threshold. These plots show the perfect agreement between the theory (circles) and the Monte-Carlo trials (solid lines) for different values of K. Figure 2 shows the threshold decrease brought by the P-AMF compared to the AMF whose theoretical distribution can be found in [1]. The threshold is closer to the optimal OGD's one for a given PFA. This explains the result obtained in figure 3 where it can be observed an improvement of about 6dB in the detection performance between AMF and P-AMF.

5. CONCLUSION

In this paper, we introduce a new adaptive detection test which takes into account the persymmetric structure of the clutter covariance matrix. This one is estimated by a Maximum Likelihood procedure. The corresponding modified Adaptive Matched Filter (AMF) is called the Persymmetric AMF. We derived the analytical distribution of the P-AMF

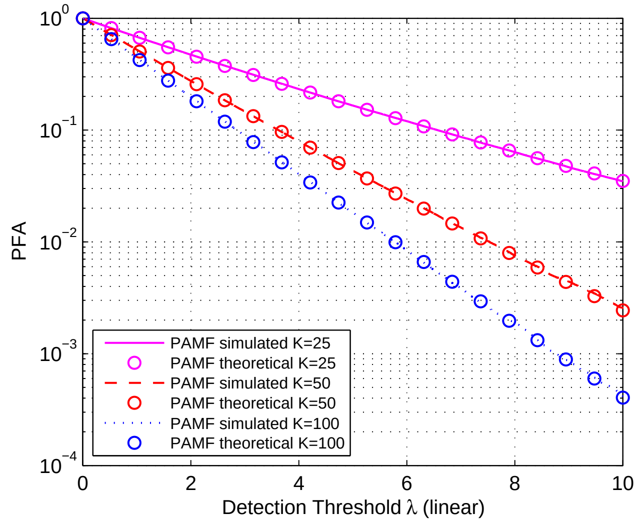

Fig. 1. PFA versus P-AMF detection threshold for different values of K: theory (circle) and Monte-Carlo (solid line).

test statistic. This result allows to set the detection threshold for a given PFA. Simulations validate theoretical results and show significant improvement of the P-AMF detection performance on the conventional AMF.

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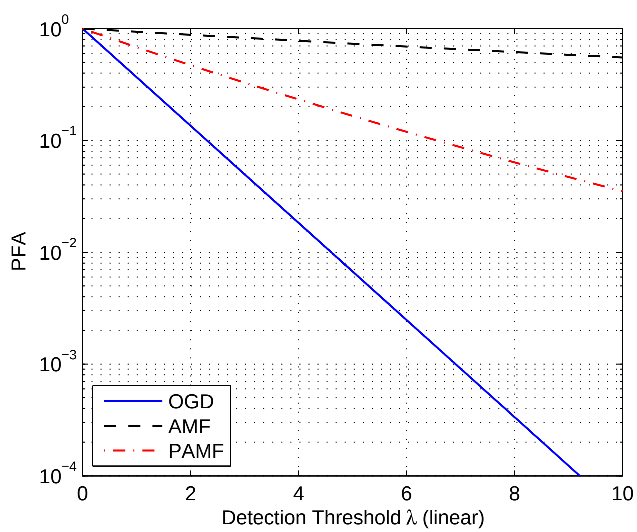

Fig. 2. PFA versus threshold for the OGD, AMF and P-AMF, for $K = 25$ and $m = 20$.

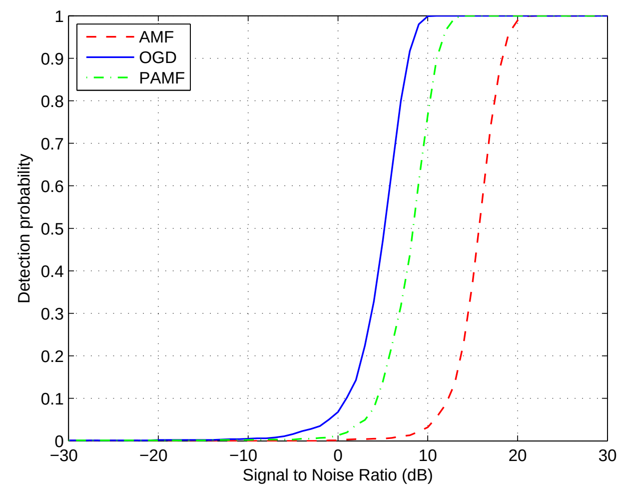

Fig. 3. Probability of detection versus Signal to Noise Ratio for $Pfa = 10^{-3}$, $K = 25$ and $m = 20$.

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