NEW METHODS OF DESIGNING OPTIMUM BROAD-BAND RADAR SIGNALS *

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1. INTRODUCTION

In radar or sonar, estimating the parameters such as the velocity or the position of a target is often a delicate problem. Let $z(t)$ be the transmitted and analytic signal with a constant propagation velocity c. The echo $x(t)$ reflected from a target moving with velocity v can be expressed as :

$$x(t,\theta) = A_0 T_\theta z(t)\, e^{i\phi} + b(t) \tag{1}$$

where T_θ is a transformation acting on the signal $z(t)$ with a vector θ of unknown parameters (time delay, Doppler shift, Doppler compression, etc..), and A_0 is the amplitude, ϕ a phase change and $b(t)$ a zero-mean white gaussian noise with σ^2 variance. When the probability density of the parameters A_0 and ϕ is unknown, the Maximum Likelihood ratio Λ to maximize, according to the Maximum Likelihood estimation theory, is given by the square modulus of the cross-ambiguity function :

$$\Lambda(\theta,\theta_0) = \frac{1}{2\sigma^2}\,\frac{\left|\displaystyle\int_{-\infty}^{+\infty} x(t,\theta)\, T^*_{\theta_0} z(t)\, dt\right|^2}{\displaystyle\int_{-\infty}^{+\infty} \left|T_{\theta_0} z(t)\right|^2\, dt} \tag{2}$$

The efficiency of an estimate $\hat\theta$ is generally measured by its variance var$(\hat\theta)$. For an unbiased estimate ($E(\hat\theta) = \theta$), this variance has a lower value given by the Cramer Rao Bounds (CRB) [1]. The CRB are obtained by inverting the Fisher Information Matrix (FIM) defined as :

$$J_{i,j} = \left(-E\left[\frac{\partial^2\Lambda}{\partial\theta_i\partial\theta_j}\right]\right)_{i,j} \tag{3}$$

where θ_i denotes each component of the vector θ.

1.1 The Narrow-Band Case

Under Woodward's conditions [2], the Doppler effect can be approximated by a shift in frequency of the signal $z(t)$. Hence, the received signal $x(t,\theta)$ can be put in the form :

$$x(t,\theta) = A_0\, z(t-\tau)\, e^{2i\pi\nu t}\, e^{i\phi} + b(t) \tag{4}$$

where $\nu = 2vf_0/c$ is the Doppler frequency shift and τ the time delay (radial position $c\tau/2$, f_0 center frequency). In this case the FIM (3) can be easily calculated and leads to :

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$$J = \frac{4\pi^2 A_0^2}{\sigma^2} \begin{pmatrix} \sigma_t^2 & f_0\, t_0 - m \\ f_0\, t_0 - m & \sigma_f^2 \end{pmatrix} \tag{5}$$

where the first order moments f_0 and t_0 represent the mean frequency and the mean epoch, and the second order moments σ_f and σ_t represent the bandwidth and the duration of the signal. The parameter m is the modulation index of the signal. Each lower bound of the variance of estimates is obtained by inverting the matrix (5). These well known results prove that the best signal in radar (good range and velocity resolutions) is characterized by a high time-bandwidth product.

1.2 The Broad-Band Case

In that case, the problem of estimating a velocity does not consist in estimating a Doppler shift but a Doppler compression factor. Thus, the echo $x(t)$ can be put in the form :

$$x(t) = A_0\, z(a_0^{-1} t - b_0)\, e^{i\phi} + b(t) \tag{6}$$

The statistic to maximize is given by the square modulus of the broad-band cross-ambiguity function which is rewritten in the frequency domain :

$$\Lambda = \frac{a}{2\sigma^2} \left| \int_0^{+\infty} X(f)\, Z^*(af)\, e^{2i\pi abf}\, df \right|^2 \tag{7}$$

where the parameters $a = (c+v)/(c-v)$ and b represent the Doppler compression and the time delay parameters to estimate. The direct calculation of the FIM (3) by classical methods is not very easy and its coefficients do not lead to a simple interpretation as in the narrow-band case [3]. In the next section and using the Mellin transform [4, 5] already used in Broad-Band signal analysis [6, 7, 9, 10], the FIM computation is easily performed and leads to a perfect physical interpretation of its coefficients in the time-frequency half plane.

2. THE FISHER INFORMATION MATRIX IN BROAD-BAND CASE

2.1 The Mellin Transform

The Mellin transform which plays an important part in the computation and the physical interpretation of the FIM's coefficients has been well defined in [4] and acts on the analytic signal $Z(f)$ in frequency by :

$$M^{\xi}[Z](\beta) = \int_0^{+\infty} Z(f)\, e^{2i\pi\xi f}\, f^{2i\pi\beta + r}\, df \tag{8}$$

This transform can be interpreted as the coefficient of the decomposition of the signal onto a hyperbolic signals basis with a group delay law given in the time-frequency half plane by the equation $t = \xi + \beta/f$ with the invariant scalar product given by :

$$\int_0^{+\infty} Z_1(f) Z_2^*(f) f^{2r+1} df = \int_{-\infty}^{+\infty} M^{\xi}[Z_1](\beta) M^{\xi *}[Z_2](\beta) d\beta \tag{9}$$

The dual Mellin variable β therefore characterizes the coefficient of an hyperbola in the time-frequency half plane. The parameter r is free but is chosen here equal to $-1/2$ to preserve the classical scalar product. The study of the tomographic construction of the unitary affine time-frequency distribution $P_0(t,f)$ [11] has shown that a signal localized in the time-frequency half plane has a Mellin transform support bounded in Mellin space (cf. figure1). The connection between the P_0 distribution :

$$P_0(t,f) = f \int_{-\infty}^{+\infty} \frac{u}{2\sinh u/2}\, Z\left(\frac{ufe^{-u/2}}{2\sinh u/2}\right) Z^*\left(\frac{ufe^{u/2}}{2\sinh u/2}\right) e^{-2i\pi f t u}\, du \tag{10}$$

and the Mellin transform is nothing but a hyperbolic Radon transform :

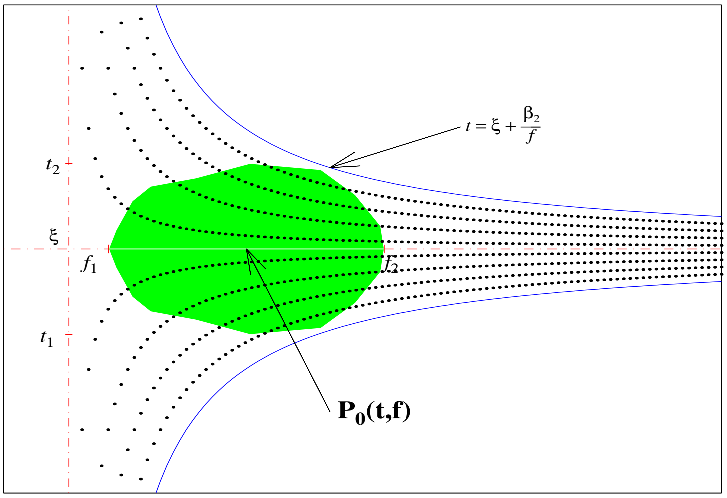

Figure 1: Localization in the time-frequency half plane of a signal $Z(f)$ having a time-frequency energy distribution $P_0(t,f)$. The two hyperbolas defined by equations $t = \xi + \beta_1/f$ and $t = \xi + \beta_2/f$ delimit the support $[\beta_1, \beta_2]$ of its Mellin transform.

$$\int_{-\infty}^{+\infty} dt \int_{0}^{+\infty} P_0(t,f)\, \delta(t - \xi - \beta/f)\, f^{-1}\, df = \left| M^{\xi}[Z](\beta) \right|^2 \tag{11}$$

Using an a priori knowledge of the localization of the signal in the time-frequency half plane (bandwidth, relative bandwidth, duration), it is now possible to perfectly determine the spread $\sigma_\beta = \beta_2 - \beta_1$ of the signal in the Mellin space (cf. figure 1). In the following, the ξ parameter will be chosen equal to zero and the transform will be noted $M[Z](\beta)$. The main property of the Mellin transform is the property of scale invariance :

$$\begin{array}{ccc} Z(f) & \longrightarrow & Z'(f) = \sqrt{a}\, Z(af) \\ \downarrow & & \downarrow \\ M[Z](\beta) & \longrightarrow & M[Z'](\beta) = a^{-2i\pi\beta} M[Z](\beta) \end{array} \tag{12}$$

which is useful when rewriting (7) :

$$\Lambda = \frac{1}{2\sigma^2} \left| \int_{-\infty}^{+\infty} M[X](\beta)\, M^*[Z_b](\beta)\, a^{2i\pi\beta}\, d\beta \right|^2 \tag{13}$$

with $Z_b(f) = Z(f)\, \exp(2i\pi bf)$. Another important property of the Mellin transform, useful for computation of the FIM coefficients, is the diagonalization of the operator B defined by :

$$BZ(f) = -\frac{1}{2i\pi} \left(f \frac{d}{df} + \frac{1}{2} \right) Z(f) \tag{14}$$

which is transformed as $M[BZ](\beta) = \beta M[Z](\beta)$

2.2 Broad-Band Expression of the Fisher Information Matrix

The Fisher Information Matrix has the form [8] :

$$J = \frac{4\pi^2 A_0^2}{\sigma^2} \begin{pmatrix} \sigma_\beta^2 & f_0\beta_0 - M \\ f_0\beta_0 - M & \sigma_f^2 \end{pmatrix} \tag{15}$$

where the parameters σ_f and f_0 define the bandwidth and the mean frequency of the signal and where the parameters β_0, σ_β are given by :

$$\beta_0 = \int_{-\infty}^{+\infty} \beta \; |M[Z](\beta)|^2 \; d\beta \qquad \sigma_\beta^2 = \int_{-\infty}^{+\infty} (\beta - \beta_0)^2 \; |M[Z](\beta)|^2 \; d\beta \tag{16}$$

The first and second order moments can be viewed respectively as the mean β and the spread of the signal Z in Mellin space. The broad-band modulation index M defined by :

$$M = -\frac{1}{2\pi} Im \int_0^{+\infty} f^2 \, \frac{dZ}{df} \, Z^*(f) \, df \tag{17}$$

plays the same role for the hyperbolic signals as the narrow-band modulation index m for the chirp signals. Finally, the ratio A_0^2/σ^2 is the Signal-to-Noise Ratio. The proof of this result can be found in annexe.

To estimate the quality of the compression and delay parameters, the FIM must be inverted. Each term of the inverse matrix J^{-1} gives the variance lower bound of each estimate. As the estimates are unbiased and efficient (high SNR), the CRB are reached and we obtain the following important new results :

- The variance of the time delay estimate $\hat{b}$ is given by :

$$\text{var}(\hat{b}) = \frac{\sigma^2}{4\pi^2 A_0^2} \frac{\sigma_\beta^2}{\sigma_f^2 \sigma_\beta^2 - (M - \beta_0 f_0)^2} \geq \frac{\sigma^2}{4\pi^2 A_0^2} \frac{1}{\sigma_f^2} \tag{18}$$

- The variance of the compression estimate $\hat{a}$ is given by:

$$\text{var}(\hat{a}) = \frac{\sigma^2}{4\pi^2 A_0^2} \frac{\sigma_f^2}{\sigma_f^2 \sigma_\beta^2 - (M - \beta_0 f_0)^2} \geq \frac{\sigma^2}{4\pi^2 A_0^2} \frac{1}{\sigma_\beta^2} \tag{19}$$

- The variance of the velocity estimate $\hat{v}$ is given by : $\text{var}(\hat{v}) = \dfrac{c^2}{4} \text{var}(\hat{a})$

- The covariance of the cross-estimates is given by :

$$\text{cov}(\hat{a}, \hat{b}) = \frac{\sigma^2}{4\pi^2 A_0^2} \frac{M - \beta_0 f_0}{\sigma_f^2 \sigma_\beta^2 - (M - \beta_0 f_0)^2} \tag{20}$$

The first result (18) shows that the time delay (or range) resolution is always related to the inverse of the signal spread in frequency as in the narrow-band case. The result (19) is very important because it proves that the compression (or velocity) resolution depends only on the inverse of the signal spread in the Mellin space instead of the signal duration as in the narrow-band case. As an example, let us consider the so-called Doppler invariant signals as hyperbolic signals (cf. figure 2) which are characterized by a no spread in Mellin space ($\sigma_\beta = 0$) : this kind of signals does not lead to a good velocity resolution (well known result). The figure 3 shows, on the contrary, that a signal with very short duration can have a no negligible velocity resolution. These two extreme examples prove the difference with the results classicaly obtained in the narrow-band case.

Under Woodward's assumptions (narrow-band case, $v/c \ll 1$), the hyperbolas which delimit the signal in the time-frequency half plane, may be replaced by straight lines parallel to the frequency axis. In that case, the parameter β, β_0, σ_β, M and $a = (c+v)/(c-v)$ can be respectively approximated by $f_0 t$, $f_0 t_0$, $f_0 \sigma_t$, $f_0 m$ and $1 + 2v/c$. Substituting these approximations in (19), we obtain the classical narrow-band results :

$$E\left[(v - \hat{v})^2\right] = \frac{c^2 \sigma^2}{16\pi^2 A_0^2 f_0^2} \frac{\sigma_f^2}{\sigma_t^2 \sigma_f^2 - (m - f_0 t_0)^2} \tag{21}$$

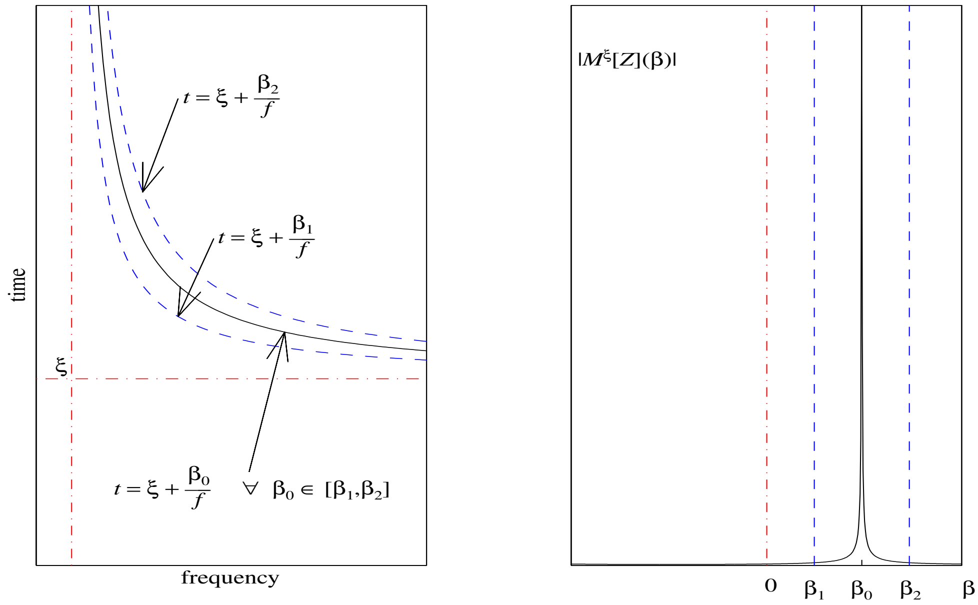

Figure 2: Localization in the time-frequency half plane of a hyperbolic signal $Z(f)$ labeled by its parameter β_0 and its Mellin transform. Any pair of hyperbolas with equation $t = \xi + \beta_1/f$ and $t = \xi + \beta_2/f$ (with $\beta_1 < \beta_0 < \beta_2$) can delimit the signal. Such a signal, although it has a infinite duration, has a zero-spread in Mellin space and therefore no velocity resolution.

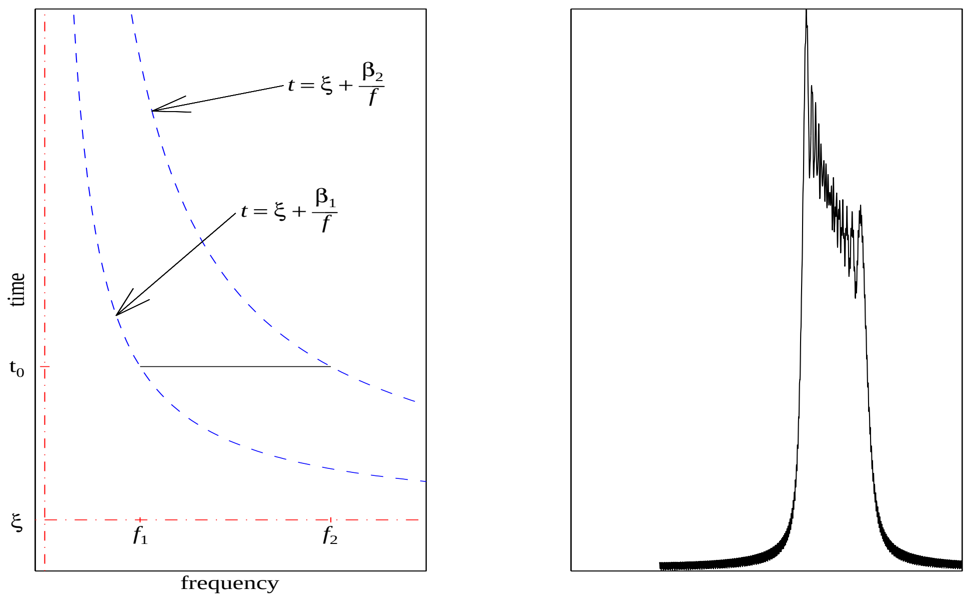

Figure 3: Localization in the time-frequency half plane of a short pulse centered around $t = t_0$ with a bandwidth $B = f_2 - f_1$ around $f_0 = (f_1 + f_2)/2$ and its Mellin transform. Such a signal, although of very short duration, has a spread $\sigma_\beta = (f_2 - f_1)t_0$ in Mellin space and therefore a finite velocity resolution.

3. HIGH $B\Delta_\beta$ SIGNALS SYNTHESIS METHODS

The two methods which will be presented are useful when looking for signals which minimize the Cramer-Rao lower bounds. The first method is devoted to the construction of optimal broad-band signals with given autocorrelation functions in velocity and delay spaces. The second one determines a phase law which allows the signal to reach the desired spreads in the Mellin and frequency spaces.

3.1 The Stationary Phase Method

The Stationary Phase Principle method already used for designing high time bandwidth product signals [12] is applied here but is extended to the Mellin and frequency spaces. The main idea is to construct high $B\Delta_\beta$ product signals (asymptotics signals) in the same way. The inverse Mellin transform is defined by :

$$Z(f) = e^{-2i\pi\xi f}\, f^{-1/2} \int_{-\infty}^{+\infty} M^{\xi}[Z](\beta)\, f^{-2i\pi\beta}\, d\beta \tag{22}$$

Following the stationary phase principle method and applying it on (22), we have :

$$Z(f) = f^{-1/2}\, e^{-2i\pi\xi f} \sqrt{\frac{2\pi}{|\phi''(\lambda)|}} \left| M^{\xi}[Z](\lambda) \right| e^{i(\phi(\lambda) - 2\pi\lambda \log f \pm \pi/4)} \tag{23}$$

where we note $M^{\xi}[Z](\beta) = \left| M^{\xi}[Z](\beta) \right| \exp\left(i\phi(\beta)\right)$ and where λ is the stationary point defined by the following equation :

$$\frac{d}{d\beta}\left[\phi(\beta) - 2\pi\beta \log f\right]_{\beta=\lambda} = 0 \tag{24}$$

or, if we note ϕ'^{-1} the reciprocal function of ϕ', defined by $\lambda = \phi'^{-1}\left(2\pi \log f\right)$.

The spectrum phase law has the form $\Psi(f) = -2\pi\xi f + \phi(\lambda) - 2\pi\lambda \log f \pm \pi/4$ and is thus defined by its group delay ($T(f) = -\frac{1}{2\pi}\frac{d\Psi}{df}$) given by $T(f) = \xi + \frac{\lambda}{f}$.

Acting on the shape of $|Z(f)|$ and $|M^{\xi}[Z](\beta)|$ by choosing the distance autocorrelation function $F(b)$ and velocity autocorrelation function $G(a)$ defined according to :

$$|Z(f)|^2 = \int_{-\infty}^{+\infty} F(b)\, e^{-2i\pi b f}\, db \qquad \left| M^{\xi}[Z](\beta) \right|^2 = \int_{0}^{+\infty} G(a)\, a^{-2i\pi\beta - 1}\, da \tag{25}$$

we thus define the phase law $\phi(\lambda)$ given by the differential equation :

$$\phi''(\lambda) = 2\pi \frac{\left| M^{\xi}[Z](\lambda) \right|^2}{f\, |Z(f)|^2} \tag{26}$$

Choosing $\psi(\lambda) = \frac{1}{2\pi}\phi'(\lambda) = \log f$, the last equation can be integrated with respect to λ and leads to :

$$\int_{0}^{\exp\psi(\lambda)} |Z(f)|^2\, df = \int_{-\infty}^{\lambda} \left| M^{\xi}[Z](\beta) \right|^2 d\beta \tag{27}$$

By choosing a given λ, it is now possible by (27) to find $\psi(\lambda)$ and to determine the phase law $\phi(\beta)$ by :

$$\phi(\beta) = 2\pi \int_{-\infty}^{\beta} \psi(u)\, du \tag{28}$$

3.2 Construction of Optimal Signals

Consider a monochromatic and analytic signal given by its equation $Z(f) = \delta(f - f_0)$. This signal has a Mellin transform given by $M^{\xi}[Z](\beta) = f_0^{2i\pi\beta - 1/2}\, e^{2i\pi\xi f_0}$. We can therefore perfectly determine the frequency law of the signal $Z(f)$ as the function of the Mellin variable :

$$f_0 = \exp\left(\frac{1}{2\pi}\frac{d\phi}{d\beta}\right) \tag{29}$$

where $\phi(\beta)$ is the phase of the Mellin transform of Z. Extending this relation, we obtain the expression of the frequency in terms of the β variable :

$$f(\beta) = \exp\left(\frac{1}{2\pi}\,\frac{d\phi(\beta)}{d\beta}\right) \tag{30}$$

Given a frequency law $f(\beta)$ in Mellin space, we can obtain by solving (30) the derivative of the Mellin phase and hence the expression of the signal in Mellin space $M^{\xi}[Z](\beta) = e^{i\phi(\beta)}$. This procedure is the analogous construction of a signal from time to frequency space using the definition of the instantaneous frequency. It only ensures that the signal will have, at one and the same time, a given bandwidth and spread in Mellin space but does not ensure the sidelobes quality of the two autocorrelation functions in range and velocity spaces.

4. CONCLUSION

The analytical expression of the Cramer Rao bounds for velocity estimation in the broad-band case has been established using the Mellin transform. The most impressive result concerns the velocity resolution of active radar (and particularly sonar) which is not related to the inverse of the signal duration as in narrow-band case but to the inverse of the spread of the signal in Mellin space. This spread has a direct geometrical interpretation in the time-frequency half plane and can be easily estimated when duration, bandwidth and relative bandwidth are known. Thanks to this interpretation, two interesting procedures have been proposed to construct optimal broad-band signals which minimize the Cramer-Rao lower bounds.

ANNEXE : Proof of the Proposition

The main idea for FIM derivation is to compute the FIM coefficients from the statistics Λ (13) rewritten in Mellin space rather than a direct computation. To simplify the demonstration, all the partial derivatives of the statistic Λ with respect to parameters a and b will be evaluated at the point O$(a = a_0 = 1, b = b_0 = 0)$. If we note $A(a, b)$, the classical cross-ambiguity function rewritten in Mellin space :

$$A(a, b) = \int_{-\infty}^{+\infty} M[X](\beta)\, M^{*}[Z_b](\beta)\, a^{2i\pi\beta}\, d\beta \tag{31}$$

all the partial derivatives of A with respect to parameters a and b and evaluated at the point O$(a = 1, b = 0)$, when using the property of unitarity of the Mellin transform (9), lead to :

$$\frac{\partial A}{\partial a} = 2i\pi \int_{-\infty}^{+\infty} \beta\, M[X](\beta)\, M^{*}[Z](\beta)\, d\beta \tag{32}$$

$$\frac{\partial^2 A}{\partial a^2} = 2i\pi \int_{-\infty}^{+\infty} \beta(2i\pi\beta - 1)\, M[X](\beta)\, M^{*}[Z](\beta)\, d\beta \tag{33}$$

$$\frac{\partial A}{\partial b} = 2i\pi \int_{-\infty}^{+\infty} M[X](\beta)\, M^{*}[fZ(f)](\beta)\, d\beta = 2i\pi \int_{0}^{+\infty} f\, X(f)\, Z^{*}(f)\, df \tag{34}$$

$$\frac{\partial^2 A}{\partial a \partial b} = -4\pi^2 \int_{-\infty}^{+\infty} \beta\, M[X](\beta)\, M^{*}[fZ(f)](\beta)\, d\beta \tag{35}$$

$$\frac{\partial^2 A}{\partial b^2} = -4\pi^2 \int_{0}^{+\infty} f^2\, X(f)\, Z^{*}(f)\, df \tag{36}$$

The first coefficient J_{11} of the FIM takes the form :

$$J_{11} = -\frac{1}{\sigma^2}\, E\left[Re\left(A^{*}\,\frac{\partial^2 A}{\partial a^2} + \left|\frac{\partial A}{\partial a}\right|^2\right)\right] \tag{37}$$

The noise $b(t)$ is a zero mean white gaussian noise. If we note $C(\beta_1 - \beta_2)$ the covariance of the Mellin transform of X, it is easy to show that :

$$C(\beta_1-\beta_2) = E\left[M[X](\beta_1) M^*[X](\beta_2)\right] = A_0^2\, M[Z](\beta_1) M^*[Z](\beta_2) + \sigma^2\, \delta(\beta_1-\beta_2) \tag{38}$$

while the covariance $C(f_1-f_2)$ of the Fourier transform of the signal is given by :

$$C(f_1-f_2) = E\left[X(f_1) X^*(f_2)\right] = A_0^2\, Z(f_1)\, Z^*(f_2) + \sigma^2\, \delta(f_1-f_2) \tag{39}$$

Using relations (32), (33) and (38), J_{11} can be rewritten :

$$J_{11} = -\frac{1}{\sigma^2} Re \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} C(\beta_1-\beta_2) \left[2i\pi\beta_1(2i\pi\beta_1 - 1) + 4\pi^2\beta_1\beta_2\right] d\beta_1\, d\beta_2 \tag{40}$$

and we obtain the expression of the J_{11} coefficient :

$$J_{11} = \frac{4\pi^2 A_0^2}{\sigma^2} \int_{-\infty}^{+\infty} (\beta-\beta_0)^2\, |M[Z](\beta)|^2\; d\beta = \frac{4\pi^2 A_0^2}{\sigma^2}\, \sigma_\beta^2 \tag{41}$$

The computation of the J_{22} coefficient of the FIM is easily performed in the same way using relations (34) and (36) and leads to :

$$J_{22} = \frac{4\pi^2}{\sigma^2}\, Re \int_0^{+\infty} \int_0^{+\infty} C(f_1-f_2)\, Z^*(f_1)\, Z(f_2)\, (f_1^2 - f_1 f_2)\, df_1\, df_2 \tag{42}$$

Substituting in (42) the covariance given by (39), we obtain the J_{22} coefficient proposed in (15). Finally, the last symmetrical coefficient J_{12} or J_{21} computation is given by :

$$J_{12} = -\frac{1}{\sigma^2}\, E\left[Re\left(X^*\, \frac{\partial^2 X}{\partial a\, \partial b} + \frac{\partial X}{\partial a}\frac{\partial X^*}{\partial b}\right)\right] \tag{43}$$

which, using relations (32), (34) and (35) and substituting the covariance given in (38), leads to :

$$\begin{aligned} J_{12} \;&=\; -\frac{4\pi^2 A_0^2}{\sigma^2} Re\left[\int_{-\infty}^{+\infty} \beta_1 M[Z](\beta_1)\, M^*[fZ(f)](\beta_1)\, d\beta_1 \right.\\ &\left. - \int_{-\infty}^{+\infty} M[Z](\beta_1)\, M^*[fZ(f)](\beta_1)\, d\beta_1 \int_{-\infty}^{+\infty} \beta_2\; |M[Z](\beta_2)|^2\; d\beta_2\right] \end{aligned} \tag{44}$$

This last expression can be easily transformed in the frequency domain using the unitarity property (9) of the Mellin transform and using the operator B defined by equation (14). With the definition of the broad-band modulation index M defined by (17), the coefficient J_{12} is finally derived.

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