Phase Code Optimization for Coherent MIMO Radar Via a Gradient Descent

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***Abstract*—In this paper, a gradient descent method is used to build radar waveform sequences with good autocorrelation and/or cross-correlation. The approach we propose is based on the energy, a function that measures the sidelobe level of a sequence, and its gradient. Then, we extend and apply it to the optimization of the coherent MIMO (Multiple Input Multiple Output) ambiguity function. We suggest to look for the transmitted signals that reduce the autocorrelation sidelobe level of the signal transmitted by the whole antenna. The obtained results, highlighted by the low sidelobe level of the ambiguity function, seem promising.**

***Index Terms*—Waveform design – Aperiodic autocorrelation – Cross-correlation – Gradient Descent – Coherent MIMO Radar – Ambiguity Function**

I. INTRODUCTION

Multiple Input Multiple Output (MIMO) radar is an emerging concept that has been inspired from wireless communications. Transmit elements of a MIMO radar send different signals which can be separated at the receiving end, providing waveform diversity. Two configurations of MIMO radars are usually considered, denoted as statistical and coherent. In a statistical MIMO radar, antenna elements are widely separated, hence improving detection performance [1]. On the other hand, all the elements of a coherent (or co-located) MIMO radar are closely spaced, providing a better spatial resolution [2]. This paper focuses on the latter, and especially on waveform design.

MIMO radar waveforms can be classified into different categories [3]: CDMA (*Code Division Multiple Access* – phase code per antenna), FDMA (*Frequency-Division Multiple Access* – one frequency per antenna), TDMA (*Time Division Multiple Access* – transmission in time), etc. However, all these signals suffer from a range/angle coupling [4]. CDMA waveforms seem to present the best coupling, at the cost of high range sidelobe levels.

In general, considered phase codes are sequences providing autocorrelation and cross-correlation with low sidelobes, because a signal with a "good" autocorrelation property may be distinguished from a time-shifted version of itself, while the cross-correlation property enables a signal to be set apart from another signal. Some known families of sequences have been reviewed in [5].

Searching for those sequences can be seen as an optimization problem, for instance the minimization of some energy criteria. The energy characterizes, for instance, the autocorrelation sidelobe level of a sequence, like the Merit Factor introduced by Golay [6]. A branch and bound approach [7] or an evolutionary strategy [8] can be used to solve this optimization problem. However, these algorithms become very expensive and ineffective with long sequences.

A recent article [9] gives a solution to the particular problem of the autocorrelation sidelobe minimization based on a steepest descent algorithm based on the gradient of the sidelobe energy. This method will be extended to the coherent MIMO radar case. Instead of optimizing on the original signals transmitted by the antennas, the optimization will rather be done on the signal transmitted in different directions. These directions are obtained by linear combinations of the signals transmitted by the antennas.

This paper is organized as follows. In Section II, a gradient descent is used in order to improve the autocorrelation of a sequence, the cross-correlation of a couple of sequences, and then both simultaneously. Section III tries to reduce the sidelobes of the coherent MIMO ambiguity function, again with a gradient descent.

II. OPTIMIZATION OF THE AUTOCORRELATION AND THE CROSS-CORRELATION OF A SIGNAL

This section introduces a set of polyphase sequences with small autocorrelations and cross-correlations. At first, each property is studied separately for one sequence for the autocorrelation, and a couple of sequences for the cross-correlation case; obtained results are then combined.

The procedure employed here is inspired from a recent article (Baden *et al.* [9]). It is based on a real function, called "energy", that quantifies the energy present in the autocorrelation sidelobes of a given sequence. It may be enough to hunt for minima of the energy function to obtain sequences with a low sidelobe level. Hence, this search is equivalent to an optimization problem.

As said in the introduction, stochastic methods are usually helpful. Their convergence to a global minimum is *almost certain* theoretically. In practice though, global convergence cannot be established, and furthermore, the longer the sequence is, the slower the algorithms are.

Here, a gradient descent, also known as a steep descent, will be used. The steep descent is a fast algorithm for finding a local minimum of a function. It starts with an initial guess of the solution, and as many times as needed, moves it towards the opposite direction of the gradient at that point. One then needs to compute the partial derivatives of each energy (autocorrelation, cross-correlation) in regards to the different phases. The derivation of the gradient of the cross-correlation is shown in section II-A, whereas the autocorrelation one is shown section II-B. Illustrations of optimizations from random polyphase sequences will be shown.

Note that steepest descent algorithms are bound to converge to local optima of the cost function. However, as will be seen thereafter, it appears that for the particular case of the phase codes, these local optima provide already quite interesting sidelobe levels.

A. Optimizing polyphase cross-correlation

This part deals with the reduction of cross-correlation sidelobes for a couple of polyphase sequences. It derives from the computation of the energy gradient, introduced by Baden *et al.* in [9]. The "energy" is a function that evaluates the sidelobes level of a sequence (here, the cross-correlation sidelobes level).

Let us consider two polyphase sequences $\mathbf{a}$ and $\mathbf{b}$, following the same hypothesis: they are of length N and of constant modulus, *i.e.* $\mathbf{a} = [a_1, ..., a_N]^T = \left(\mathrm{e}^{\jmath 2\pi\alpha_i}\right)_{i\in[\![1,N]\!]}$. Assume that $a_i = 0$ for $i < 1$ and $i > N$.

The discrete cross-correlation, denoted by $\mathbf{x} = \mathbf{a} * \mathbf{b}$, is the sequence:

$$x_m = \sum_{i=1}^{N} a_i b_{i+m}^* \quad (-N < m < N)\,. \tag{1}$$

By including some weighting $\mathbf{w}$ to allow shaping of the sidelobes, and an exponent p to control the peak sidelobe, the cross-correlation sidelobe energy E_c is defined by:

$$E_c(a,b) = \sum_{m=-N+1}^{N-1} w_m \left(x_m x_m^*\right)^p\,. \tag{2}$$

As a small energy means low sidelobes, a steepest descent should find a "better" sequence. However, it requires a gradient computation; the procedure is detailed in [9], and main results are given below.

In the following, we note $(.)^*$ the complex conjugate operator, $(.)^r$ the reverse part of any vector (*i.e.* $\mathbf{a}^r = [a_N, \ldots, a_1]$), the real part and imaginary part are respectively denoted by $\Re(.)$ and $\Im(.)$.

According to the chain rule, the gradient of E_c with respect to the phase angle α can be written as:

$$\frac{\partial E_c}{\partial \alpha_j} = \Re(a_j)\frac{\partial E_c}{\partial \Im(a_j)} - \Im(a_j)\frac{\partial E_c}{\partial \Re(a_j)}\,. \tag{3}$$

So we have to examine the partial derivatives of E_c with respect to the real and imaginary parts of $\mathbf{a}$. Let us set $\boldsymbol{\gamma} = (\gamma_{-N+1}, \ldots, \gamma_{N-1})^T$ where $\gamma_m = w_m(c_m\, c_m^*)^{p-1}$. These partial derivatives are:

$$\begin{aligned}\frac{\partial E_c}{\partial \Re(a_j)} &= 2p \sum_{m=-N+1}^{N-1} \gamma_m \Big[\Re(x_m)\Re(b_{j+m}) \\ &\qquad - \Im(x_m)\Im(b_{j+m})\Big] \\ &= 2p\,\Re[((\gamma\circ\mathbf{x}) * \mathbf{b}^*)_j]\,, \\ \frac{\partial E_c}{\partial \Im(a_j)} &= 2p \sum_{m=-N+1}^{N-1} \gamma_m \Big[\Re(x_m)\Im(b_{j+m}) \\ &\qquad + \Im(x_m)\Re(b_{j+m})\Big] \\ &= 2p\,\Im[((\gamma\circ\mathbf{x}) * \mathbf{b}^*)_j]\,.\end{aligned} \tag{4, 5}$$

where $\circ$ denotes the Hadamard product. Putting (4) and (5) into the equation (3) gives:

$$\begin{aligned}\frac{\partial E_c}{\partial \alpha_j} &= 2p\,\Re(a_j)\Im[((\gamma\circ\mathbf{x}) * \mathbf{b}^*)_j] \\ &\qquad - 2p\,\Im(a_j)\Re[((\gamma\circ\mathbf{x}) * \mathbf{b}^*)_j] \\ &= 2p\,\Im\left[a_j^*((\gamma\circ\mathbf{x}) * \mathbf{b}^*)_j\right]\,.\end{aligned} \tag{6}$$

In the same way, the partial derivative of the cross-correlation energy with respect to β – the phase angle of $\mathbf{b}$ – should be computed. As the calculation steps are quite analogous, the main results are listed below.

$$\begin{aligned}\frac{\partial E_c}{\partial \Re(b_j)} &= 2p\,\Re[(\gamma\circ\mathbf{x}) * \mathbf{a}^{*r}]_{N+1-j}\,, \\ \frac{\partial E_c}{\partial \Im(b_j)} &= -2p\,\Im[(\gamma\circ\mathbf{x}) * \mathbf{a}^{*r}]_{N+1-j}\,, \\ \frac{\partial E_c}{\partial \beta_j} &= -2p\,\Im\left[b_j((\gamma\circ\mathbf{x}) * \mathbf{a}^{*r})_{N+1-j}\right]\,.\end{aligned} \tag{7}$$

Finally, an optimization of $\mathbf{a}$ and $\mathbf{b}$ can be done, again with a gradient descent, in order to find a couple of sequences with a better cross-correlation property. Figure 1 gives an example of an optimization of the cross-correlation of two sequences of length $N = 256$. Both sequences have been randomly initialized; their cross-correlation sequence is represented by the red-dotted line. An enhancement of around 25 dB can be obtained after applying the previous procedure.

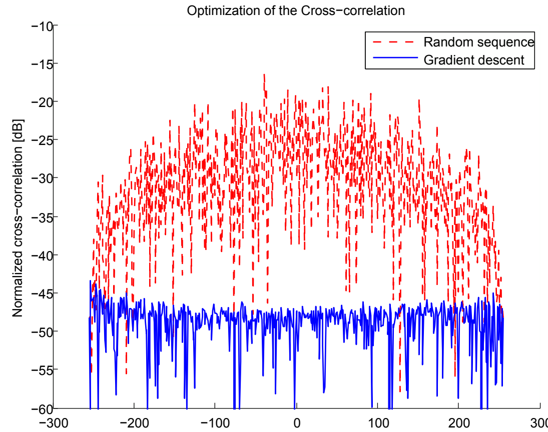

Figure 1. Optimization of the cross-correlation of a couple of sequences

B. Optimizing polyphase autocorrelation

Consider now only one sequence $\mathbf{a}$ under the same assumptions as above. The autocorrelation is expressed as $\mathbf{c} = \mathbf{a} * \mathbf{a}$ and can be computed as:

$$c_m = \sum_{i=1}^{N} a_i a_{i+m}^* \quad (-N < m < N) \tag{8}$$

The autocorrelation sidelobe energy is defined as:

$$E_a(a) = \sum_{m=-N+1}^{N-1} w_m (c_m c_m^*)^p \tag{9}$$

The gradient of this energy is needed to reduce the autocorrelation with a steepest descent. Every calculation step can be found in [9], so only major developments are listed here.

As the procedure is similar, let us start with the partial derivatives with respect to the real and imaginary parts of $\mathbf{a}$. $\boldsymbol{\eta}$ is set to $\boldsymbol{\eta}_m = w_m (c_m c_m^*)^{p-1}$ for convenience.

$$\begin{aligned}
\frac{\partial E_a}{\partial \Re(a_j)} &= 2p \sum_{m=-N+1}^{N-1} \boldsymbol{\eta}_m \Big[\Re(c_m) \left(\Re(a_{j+m}) + \Re(a_{j-m}) \right) \\
&\qquad + \Im(c_m) \left(-\Im(a_{j+m}) + \Im(a_{j-m}) \right) \Big] \\
&= 2p \, \Re \left[((\boldsymbol{\eta} \circ \mathbf{c}) * \mathbf{a}^*)_j + ((\boldsymbol{\eta} \circ \mathbf{c}) * \mathbf{a}^r)_{N+1-j} \right] , \\
\frac{\partial E_a}{\partial \Im(a_j)} &= 2p \sum_{m=-N+1}^{N-1} \boldsymbol{\eta}_m \Big[\Re(c_m) \left(\Im(a_{j+m}) + \Im(a_{j-m}) \right) \\
&\qquad + \Im(c_m) \left(\Re(a_{j+m}) - \Re(a_{j-m}) \right) \Big] \\
&= 2p \, \Im \left[((\boldsymbol{\eta} \circ \mathbf{c}) * \mathbf{a}^*)_j - ((\boldsymbol{\eta} \circ \mathbf{c}) * \mathbf{a}^r)_{N+1-j} \right] .
\end{aligned}$$

The gradient of E_a with respect to the phase angle α can be deduced with the chain rule:

$$\begin{aligned}
\frac{\partial E_a}{\partial \alpha_j} &= \Re(a_j) \frac{\partial E_a}{\partial \Im(a_j)} - \Im(a_j) \frac{\partial E_a}{\partial \Re(a_j)} \\
&= -2p \, \Im \left[a_j ((\boldsymbol{\eta} \circ \mathbf{c}^*) * \mathbf{a})_j \right] \\
&\qquad - 2p \, \Im \left[a_j ((\boldsymbol{\eta} \circ \mathbf{c}) * \mathbf{a}^r)_{N+1-j} \right] .
\end{aligned} \tag{10}$$

Figure 2 shows an illustration of an autocorrelation optimization using a steep descent. A random initialization is used. Its autocorrelation is represented with a red dotted line. A gain of 15 dB can be observed for this sequence of length $N = 256$. One can notice that the gain is smaller than the cross-correlation one; it can be explained by the increase of the number of degrees of freedom.

C. Optimizing both autocorrelation and cross-correlation

In general, optimization of the autocorrelation or the cross-correlation alone is not sufficient. It is usually desired to design a set of sequences with low sidelobes level, in both properties. This section investigates the two-sequences case, but can easily be generalized for a set of several sequences.

Here, the energy E should depict the autocorrelation and the cross-correlation of two polyphase sequences $\mathbf{a}$ and $\mathbf{b}$:

$$E = E_a(a) + E_a(b) + E_c(a, b) . \tag{11}$$

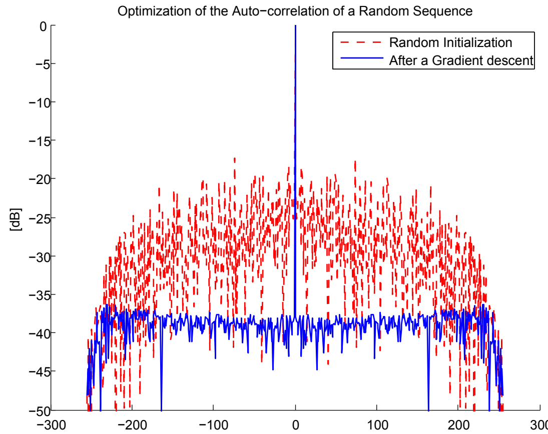

Figure 2. Optimization of the autocorrelation of a polyphase sequence

It is obvious that $E_a(a)$ is independant of $\mathbf{b}$. Gathering equations (6), (7) and (10) gives:

$$\begin{aligned}
\frac{\partial E}{\partial \alpha_j} &= \frac{\partial E_a(a)}{\partial \alpha_j} + \frac{\partial E_c(a,b)}{\partial \alpha_j} \\
&= -2p \, \Im \left[a_j ((\boldsymbol{\eta}^{\mathbf{a}} \circ \mathbf{c}^{\mathbf{a}*}) * \mathbf{a})_j \right] \\
&\qquad - 2p \, \Im \left[a_j ((\boldsymbol{\eta}^{\mathbf{a}} \circ \mathbf{c}^{\mathbf{a}}) * \mathbf{a}^r)_{N+1-j} \right] \\
&\qquad + 2p \, \Im \left[a_j^* ((\gamma \circ \mathbf{x}) * \mathbf{b}^*)_j \right] , \\
\frac{\partial E}{\partial \beta_j} &= \frac{\partial E_a(b)}{\partial \beta_j} + \frac{\partial E_c(a,b)}{\partial \beta_j} \\
&= -2p \, \Im \left[b_j ((\boldsymbol{\eta}^{\mathbf{b}} \circ \mathbf{c}^{\mathbf{a}*}) * \mathbf{b})_j \right] \\
&\qquad - 2p \, \Im \left[b_j ((\boldsymbol{\eta}^{\mathbf{b}} \circ \mathbf{c}^{\mathbf{a}}) * \mathbf{b}^r)_{N+1-j} \right] \\
&\qquad - 2p \, \Im \left[b_j ((\gamma \circ \mathbf{x}) * \mathbf{a}^{*r})_{N+1-j} \right]
\end{aligned}$$

where $\mathbf{c}^{\mathbf{a}}$ stands for the discrete autocorrelation sequence of $\mathbf{a}$, and $\boldsymbol{\eta}_m^{\mathbf{a}} = w_m (c_m^{\mathbf{a}} (c_m^{\mathbf{a}})^*)^{p-1}$.

Figure 3 depicts the autocorrelation and the cross-correlation of two sequences of length $N = 256$ after a joint optimization. The gain in autocorrelation and in cross-correlation sidelobe levels is reduced compared to the results obtained with separate optimizations. However, a compromise can be observed: each property has pretty much the same level (the peak sidelobe level is around -28 dB).

III. Coherent MIMO Waveform Design with a Gradient Descent

In the coherent MIMO Radar concept, several waveforms are sent simultaneously by different radiating elements. The radiated signal (by the whole antenna) is a linear combination of all the transmitted ones; it is distinct for each angular direction through phase-shifting in space.

The literature commonly supposes that the transmitted waveforms are orthogonal. Such orthogonality would provide the ability to separate perfectly each component in reception. Thus desired phase codes should ideally present perfect aperiodic autocorrelation and cross-correlation properties.

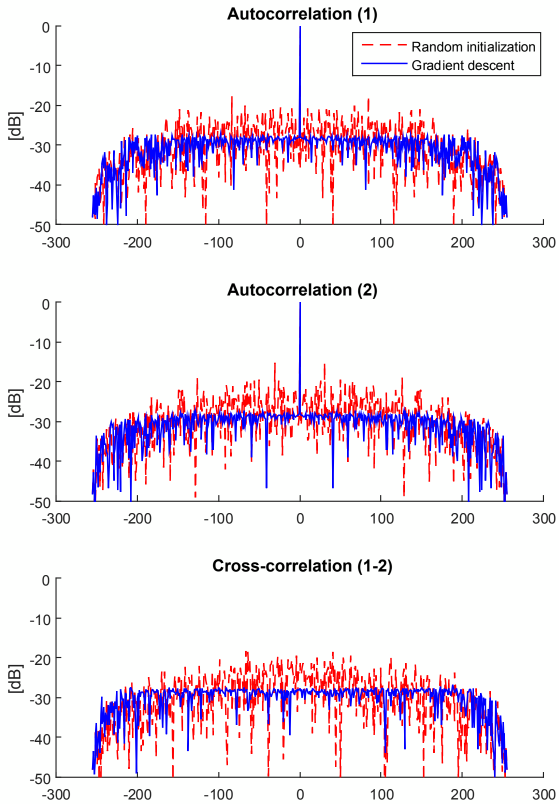

Figure 3. On the top: optimization of the autocorrelation. On the bottom: optimization of the cross-correlation

As these properties are not theoretically feasible, one should search for the best possible sequences in terms of output MIMO ambiguity sidelobes. A first solution would be to optimize the autocorrelation and the cross-correlation sidelobes of the sequences transmitted by the antennas. However, it appears that the range cut of the MIMO ambiguity function is provided by a linear combination of all autocorrelation and cross-correlation sidelobes of the transmitted sequences, with linear coefficients depending on the considered direction. It is thus preferable to optimize directly the autocorrelation and cross-correlation sidelobes of the signal transmitted in different considered directions. We propose to study the autocorrelation in several directions.

A. Problem Formulation

Let us consider a transmitting array of N_E antennas and a receiving array of N_R antennas. The signal radiated by the whole antenna, denoted by $\mathbf{s}(\theta_c)$, is [4]:

$$s_i(\theta_c) = \sum_{m=0}^{N_E-1} \mathrm{e}^{j\mathbf{x}_{E,m}^T \mathbf{k}(\theta_c)} s_i^m \,, \tag{12}$$

where:

- $\mathbf{x}_{E,m}^T$ is the position of the m^{th} antenna,
- $\mathbf{k}(\theta_c)$ is the wave vector in the direction θ_c,
- $\mathbf{s}^m := [s_1^m, ..., s_N^m]$ is the waveform of length N assigned to the m^{th} antenna.

The cross-correlation $\mathbf{x}$ between two signals at different angles θ_c and θ_c' can be written as:

$$\begin{aligned} x_l &= \sum_{i=1}^{N} s_i (s'_{i+l})^* \\ &= \sum_{i=1}^{N} \left(\sum_{m=0}^{N_E-1} \mathrm{e}^{jx_{E,m}^T k(\theta_c)} s_i^m \right) \left(\sum_{m'=0}^{N_E-1} \mathrm{e}^{jx_{E,m'}^T k(\theta_c')} s_{i+l}^{m'} \right)^* \\ &= \sum_{m,m'} \mathrm{e}^{jx_{E,m}^T k(\theta_c) - jx_{E,m'}^T k(\theta_c')} \sum_{i=1}^{N} s_i^m (s_{i+l}^{m'})^* \,. \end{aligned}$$

One can notice that the cross-correlation is an accumulation of all the autocorrelation and the cross-correlation between two transmitted signals, within a phase shift. We recall the phase shift is $\varphi_{m,m'} = \mathbf{x}_{E,m}^T \mathbf{k}(\theta_c) - \mathbf{x}_{E,m'}^T \mathbf{k}(\theta_c')$. If $\theta = \theta_c$, this reduces to the autocorrelation function.

In the same way as section II, let us define the cross-correlation sidelobe energy function:

$$E_c(s, s') = \sum_{l=-N+1}^{N-1} w_l (x_l x_l^*)^p \,.$$

This energy function shall be minimized so that the waveform transmitted by the whole antenna has low cross-correlation sidelobes in the direction θ_c. This optimization problem will also be solved by a gradient descent: the computation of the partial derivatives of the energy function with respect to the phase angle of each transmitted signal is explained in the next part.

B. Energy Gradient Calculation

The energy gradient is computed in two steps. First, partial derivatives with respect to the real and the imaginary part of s^m (the signal transmitted by each antenna) are determined. The energy gradient with respect to each phase is then deduced from the chain rule. Some developments are superfluous and are not given here.

Let us start by giving some details on the real part (and the imaginary one) of the cross-correlation sequence $\mathbf{x}$:

$$\begin{aligned} \Re(x_l) &= \sum_{m,m'} \Re \left(\mathrm{e}^{j\varphi_{m,m'}} \sum_{i=1}^{N} s_i^m (s_{i+l}^{m'})^* \right) \\ &= \sum_{m,m'} \Re(\mathrm{e}^{j\varphi_{m,m'}}) \sum_{i=1}^{N} \left[\Re(s_i^m)\Re(s_{i+l}^{m'}) + \Im(s_i^m)\Im(s_{i+l}^{m'}) \right] \\ &\quad - \sum_{m,m'} \Im(\mathrm{e}^{j\varphi_{m,m'}}) \sum_{i=1}^{N} \left[\Im(s_i^m)\Re(s_{i+l}^{m'}) - \Re(s_i^m)\Im(s_{i+l}^{m'}) \right] , \\ \Im(x_l) &= \sum_{m,m'} \Im(\mathrm{e}^{j\varphi_{m,m'}}) \sum_{i=1}^{N} \left[\Re(s_i^m)\Re(s_{i+l}^{m'}) + \Im(s_i^m)\Im(s_{i+l}^{m'}) \right] \\ &\quad + \sum_{m,m'} \Re(\mathrm{e}^{j\varphi_{m,m'}}) \sum_{i=1}^{N} \left[\Im(s_i^m)\Re(s_{i+l}^{m'}) - \Re(s_i^m)\Im(s_{i+l}^{m'}) \right] . \end{aligned}$$

As the radiated signal $\mathbf{s}(\theta_c)$ is basically a linear combination of the transmitted signals s^m, it is not absurd to consider the partial derivatives of the energy *w.r.t.* the real part (and the imaginary part) of s^m. Before that, we derive the cross-correlation $\mathbf{x}$:

$$\begin{aligned}
\frac{\partial \Re(x_l)}{\partial \Re(s_v^u)} &= \sum_{m,m'} \Re(\mathrm{e}^{j\varphi_{m,m'}}) \Big[\delta_{um}\, \Re(s_{v+l}^{m'}) + \delta_{um'}\, \Re(s_{v-l}^{m})\Big] \\
&\quad - \sum_{m,m'} \Im(\mathrm{e}^{j\varphi_{m,m'}}) \Big[- \delta_{um}\, \Im(s_{v+l}^{m'}) + \delta_{um'}\, \Im(s_{v-l}^{m})\Big] \\
&= \sum_{m,m'} \delta_{um}\, \Re(\mathrm{e}^{-j\varphi_{m,m'}} s_{v+l}^{m'}) + \delta_{um'}\, \Re(\mathrm{e}^{j\varphi_{m,m'}} s_{v-l}^{m}) \\
&= \sum_{m} \Re(\mathrm{e}^{-j\varphi_{u,m}} s_{v+l}^{m}) + \Re(\mathrm{e}^{j\varphi_{m,u}} s_{v-l}^{m})\,, \\
\frac{\partial \Im(x_l)}{\partial \Re(s_v^u)} &= \sum_{m} \Im(\mathrm{e}^{j\varphi_{u,m}} (s_{v+l}^{m})^*) + \Im(\mathrm{e}^{j\varphi_{m,u}} s_{v-l}^{m}) \\
\frac{\partial \Re(x_l)}{\partial \Im(s_v^u)} &= \sum_{m} \Im(\mathrm{e}^{-j\varphi_{u,m}} s_{v+l}^{m}) + \Im(\mathrm{e}^{j\varphi_{m,u}} s_{v-l}^{m}) \\
\frac{\partial \Im(x_l)}{\partial \Im(s_v^u)} &= \sum_{m} \Re(\mathrm{e}^{j\varphi_{u,m}} (s_{v+l}^{m})^*) - \Re(\mathrm{e}^{j\varphi_{m,u}} s_{v-l}^{m})\,.
\end{aligned}$$

Set $\boldsymbol{\gamma}$ to $\gamma_l = w_l(x_l x_l^*)^{p-1}$. Gathering all the previous results gives:

$$\begin{aligned}
\frac{\partial E_c}{\partial \Re(s_v^u)} &= 2p \sum_{l=1-N}^{N-1} \gamma_l \left(\Re(x_l) \frac{\partial \Re(x_l)}{\partial \Re(s_v^u)} + \Im(x_l) \frac{\partial \Im(x_l)}{\partial \Re(s_v^u)} \right) \\
&= 2p \sum_{l=1-N}^{N-1} \Bigg(\gamma_l\, \Re(x_l) \sum_m \Big[\Re(\mathrm{e}^{-j\varphi_{u,m}} s_{v+l}^{m}) \\
&\qquad\qquad + \Re(\mathrm{e}^{j\varphi_{m,u}} s_{v-l}^{m}) \Big] \\
&\qquad + \gamma_l\, \Im(x_l) \sum_m \Big[\Im(\mathrm{e}^{j\varphi_{u,m}} (s_{v+l}^{m})^*) \\
&\qquad\qquad + \Im(\mathrm{e}^{j\varphi_{m,u}} s_{v-l}^{m}) \Big] \Bigg) \\
&= 2p \sum_m \sum_{l=1-N}^{N-1} \Re(\gamma_l x_l \mathrm{e}^{-j\varphi_{u,m}} s_{v+l}^{m}) \\
&\qquad + \Re(\gamma_l x_l^* \mathrm{e}^{j\varphi_{m,u}} s_{v-l}^{m}) \\
&= 2p \sum_m \Re \left[((\gamma \circ x) \mathrm{e}^{-j\varphi_{u,m}} * (s^m)^*)_v \right] \\
&\qquad + \Re \left[((\gamma \circ x^*) \mathrm{e}^{j\varphi_{m,u}} * (s^m)^{*r})_{N+1-v} \right]\,, \\
\frac{\partial E_c}{\partial \Im(s_v^u)} &= 2p \sum_{l=1-N}^{N-1} \gamma_l \left(\Re(x_l) \frac{\partial \Re(x_l)}{\partial \Im(s_v^u)} + \Im(x_l) \frac{\partial \Im(x_l)}{\partial \Im(s_v^u)} \right) \\
&= 2p \sum_m \Im \left[((\gamma \circ x) \mathrm{e}^{-j\varphi_{u,m}} * (s^m)^*)_v \right] \\
&\qquad + \Im \left[((\gamma \circ x^*) \mathrm{e}^{j\varphi_{m,u}} * (s^m)^{*r})_{N+1-v} \right]\,.
\end{aligned}$$

Let $\alpha_u(v)$ the phase angle of the element s_v^u. According to the chain rule, the partial derivative of E_c with respect to that phase can be written as:

$$\begin{aligned}
\frac{\partial E_c}{\partial \alpha_u(v)} &= \Re(s_v^u) \frac{\partial E_c}{\partial \Im(s_v^u)} - \Im(s_v^u) \frac{\partial E_c}{\partial \Re(s_v^u)} \\
&= -2p \textstyle\sum_m \Big(\Im \left[s_v^u ((\gamma \circ x^*) \mathrm{e}^{j\varphi_{u,m}} * s^m)_v \right] \\
&\qquad + \Im \left[s_v^u ((\gamma \circ x) \mathrm{e}^{-j\varphi_{m,u}} * (s^m)^r)_{N+1-v} \right] \Big)\,.
\end{aligned} \tag{13}$$

C. Application and Results

In this part, we will search for transmitted sequences s^m in such a way that the aperiodic correlation of the signal produced by the array $\mathbf{s}(\theta_c)$ is optimized. Since that signal $\mathbf{s}(\theta_c)$ depends on the direction θ_c (*cf.* equation (12)), and since we want good sidelobes in any direction, we have to examine another formulation of the minimization problem that considers several directions Θ:

$$\min_{s^0,\ldots,s^{N_E-1}} E(\mathbf{s}) := \sum_{\theta\in\Theta} E_c(s(\theta), s(\theta))\,. \tag{14}$$

As ∇E can easily be deduced from eq. (13), a steepest descent is employed to find a transmitting set $\{s^0, \ldots, s^{N_E-1}\}$ (which is a local minima of E).

Figures 4 to 7 present some results of a simulated radar antenna with four transmitters ($N_E = 4$) and four receivers. Phase codes are of length $N = 256$, and four directions are inspected. Two methods are compared:

1) the proposed method, *i.e.* an optimization of the transmitted waveforms in order to improve the autocorrelation property of the radiated signal
2) the usual method, *i.e.* an optimization of the autocorrelation and the cross-correlation of the transmitting set. This set is obtained with the procedure described in Section II.

An improvement of the autocorrelation of the radiated signal in a direction of interest is shown in Figure 4. A gain of 10 dB can be noticed compared to a random initialization (represented by the red dotted line), whereas the second method does not improve this autocorrelation. In the three other directions of interest, sidelobe levels are quite similar. Moreover, Figure 5 shows that it is not needed to take into account the orthogonality of the transmitted sequences in order to improve the autocorrelation of the radiated signal in a direction.

Let us recall the problematic described in the introduction. The aim of this article is to search for phase-coded waveforms so that the radar has a good range/angle resolution, and low sidelobe levels (in range and in angle). The so-called ambiguity function can provide directly a measure of these criteria. Figure 6 and Figure 7 represent the ambiguity function obtained for the proposed method and the usual one respectively. We have made the assumption that there is no Doppler effect during a pulse. Results are promising, as the sidelobe level is around -28 dB. They are slightly better than what we get with a Gold code, [4] or with the usual method. Notice that the range cut at $\theta = 0$ is equivalent to the autocorrelation of the signal diffused by the whole antenna (*cf.* Figure 4).

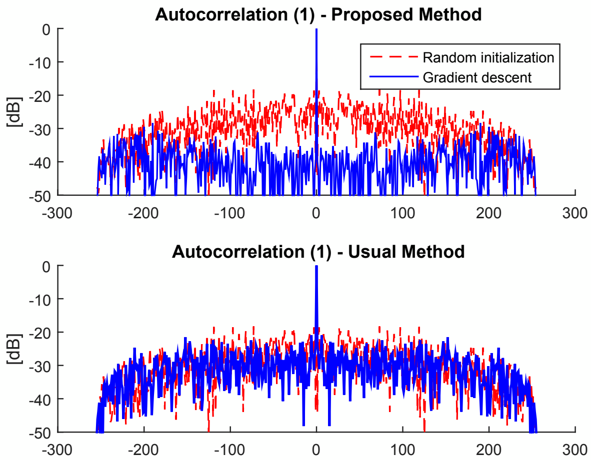

Figure 4. Optimization of the aperiodic correlation of the radiated signal in a direction of interest. On the top: with the proposed method. On the bottom: with the usual method.

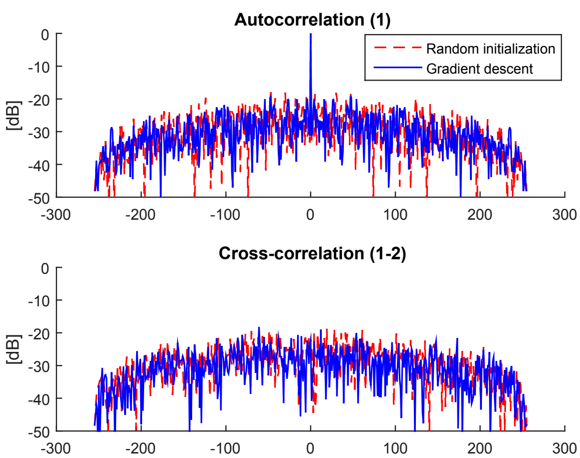

Figure 5. On top: autocorrelation of the first transmitted sequence. On the bottom: cross-correlation between two transmitted sequences.

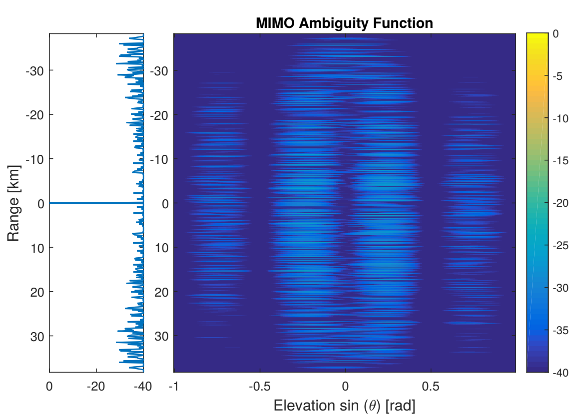

Figure 6. Ambiguity function obtained after a gradient descent (proposed method)

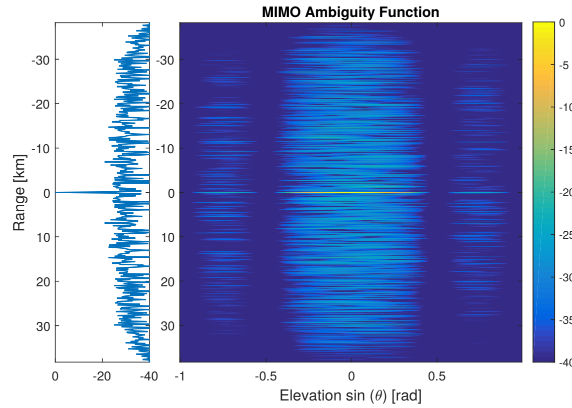

Figure 7. Ambiguity function obtained after a gradient descent (usual method)

IV. Conclusion

In this paper, we have seen that a gradient descent is an interesting method for finding sequences with low sidelobes in autocorrelation (or in cross-correlation). This method has also been applied to search for transmitted sequences of a coherent MIMO radar in a way that the emitted signal has a nice autocorrelation. Results are promising as the associated ambiguity function presents low sidelobes in range and in angle.

Ongoing work will be focused on:

- Taking into account the cross-correlation in the MIMO waveform optimization
- Optimizing the radiated energy in the different directions
- Jointly optimizing the code and its optimal mismatched filter

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