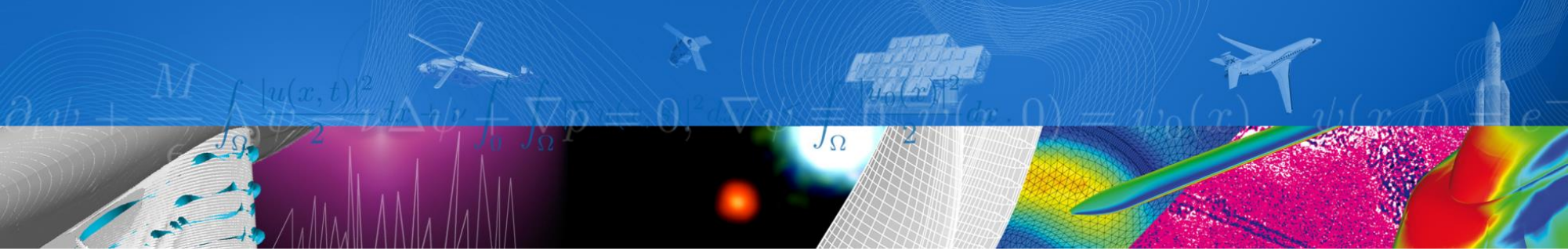

Adaptive subspace detectors for off-grid mismatched targets

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Introduction

Goal: Detection under signal mismatch has been investigated for various forms (e.g. [DeMaio 05]), fewer work concern the very structred off-grid mismatch.

Outline:

- The adaptive detection problem
- The off-grid mismatch isssue : state of the art and proposed Subspace detecor based solution.
- Adaptive subspace detector peformance
- Relevant choice of the subspace
- Numerical results

The adaptive detection problem (in radar)

- Common adaptive detection problem: binary Hypothesis test between presence ($\mathcal{H}_1$) or absence ($\mathcal{H}_0$) of a signal in noise

$$\mathcal{H}_1 : \begin{cases} \mathbf{y} = \alpha \mathbf{s}(\theta) + \mathbf{n}, & , \\ \mathbf{y}_k = \mathbf{n}_k, & 1 \leq k \leq K, \end{cases} \quad \mathcal{H}_0 : \begin{cases} \mathbf{y} = \mathbf{n}, & \\ \mathbf{y}_k = \mathbf{n}_k, & 1 \leq k \leq K. \end{cases}$$

  - $\mathbf{y} \in \mathbb{C}^{P \times 1}$: observation vector.
  - $\alpha \in \mathbb{C}$ is a deterministic, unknown (nuisance) parameter.
  - θ is the unknown signal parameter under test.
  - noise: $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R})$. In adaptive context $\mathbf{R}$ is unknown.
  - secondary data $\mathbf{y}_k = \mathbf{n}_k (1 \leq k \leq K)$ are needed to estimate $\mathbf{R}$.

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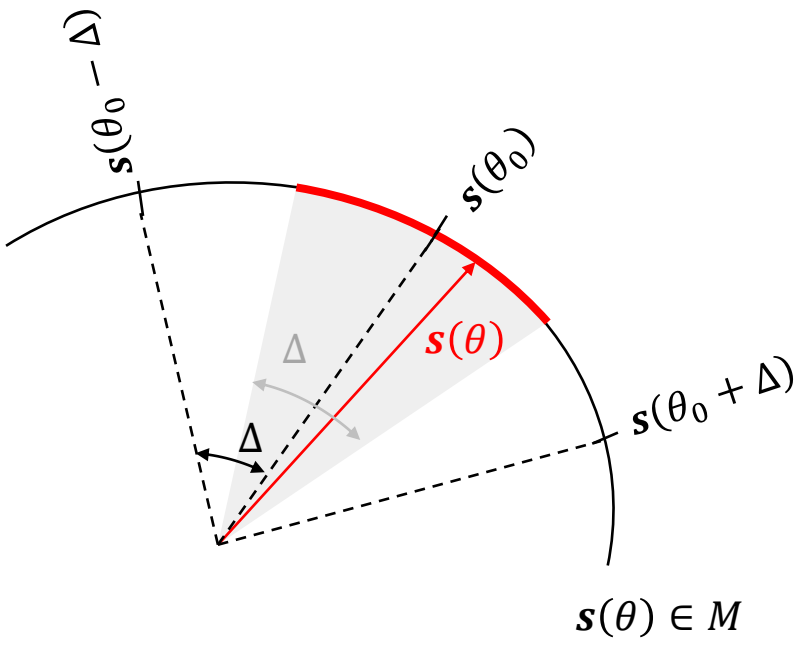
  - secondary data $\mathbf{y}_k = \mathbf{n}_k (1 \leq k \leq K)$ are needed to estimate $\mathbf{R}$.
- Common solutions :
  - Kelly's detector [Kelly 86], AMF [Robey et al. 12] and ACE [Sharf et al. 96]
- Common signal model (sinusoid in noise, radar detection after MF):

$$\mathbf{s}(\theta) = \frac{1}{\sqrt{P}} \left[\begin{array}{cccc} 1 & e^{j2\pi\theta} & \dots & e^{j2\pi(P-1)\theta} \end{array} \right]^T,$$

The off-grid issue

- As θ is unknown, the parameter space is often described by a discrete grid $\mathcal{G}_\Delta$ of step Δ.
- Δ is often selected as the main lobe width of $s(\theta)$ (sinusoids in noise $\Delta = \frac{1}{P}$):
  - two closer contributors cannot be well separated.
  - $s(\theta)$ and $s(\theta + \Delta)$ are (sometimes nearly) orthogonal.



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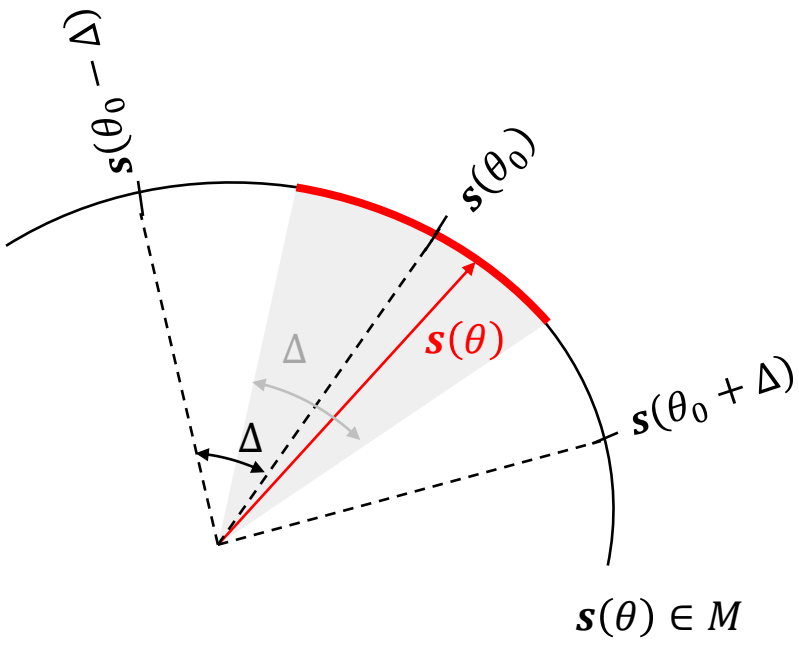
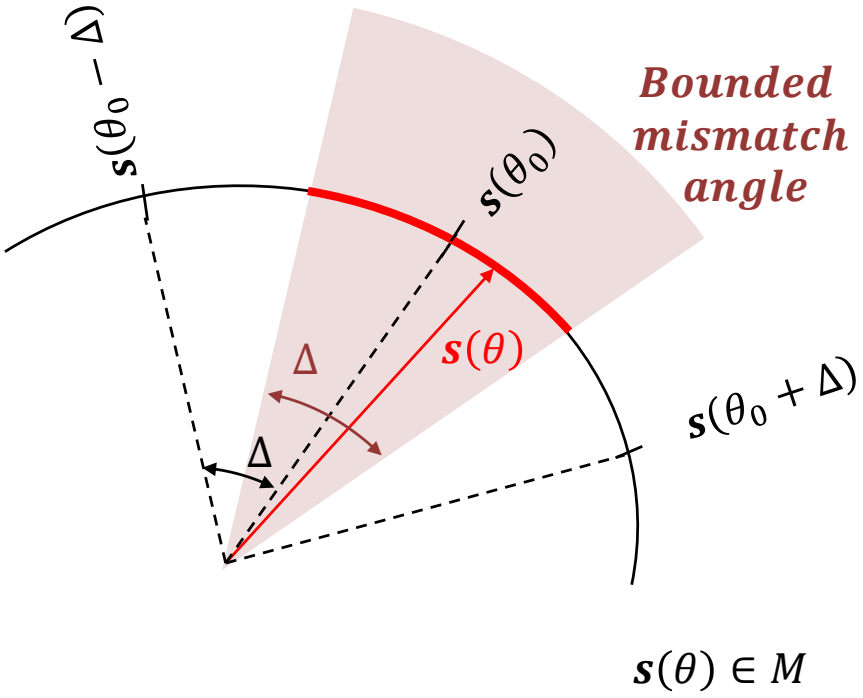


There is no reason for $\theta \in \mathcal{G}_\Delta$: we assume that θ is spread uniformly in a resolution cell $\left[\theta_0 - \frac{\Delta}{2}, \theta_0 + \frac{\Delta}{2}\right]$, $\theta_0 \in \mathcal{G}_\Delta$

- $\theta \notin \mathcal{G}_\Delta$ leads to a mismatch :
  - For normalized matched filter, the loss can be severe [Rabaste et al. 16].

The bounded angle mismatch GLRT

- The signal is assumed to lie in a cone $\mathbf{s} \in \mathscr{C}_{\theta_c} = \left\{\mathbf{s}; \frac{\mathbf{s}^H \mathbf{P}_\mathbf{A} \mathbf{s}}{\mathbf{s}^H \mathbf{s}} \leq cos^2(\theta_c)\right\}$, where $\mathbf{P}_\mathbf{A}$ is the projector on $\mathbf{A}$.
- The GLRT derived in [Besson 06]
- Suited to more general mismatch.
- No theoretic expression of the performances.



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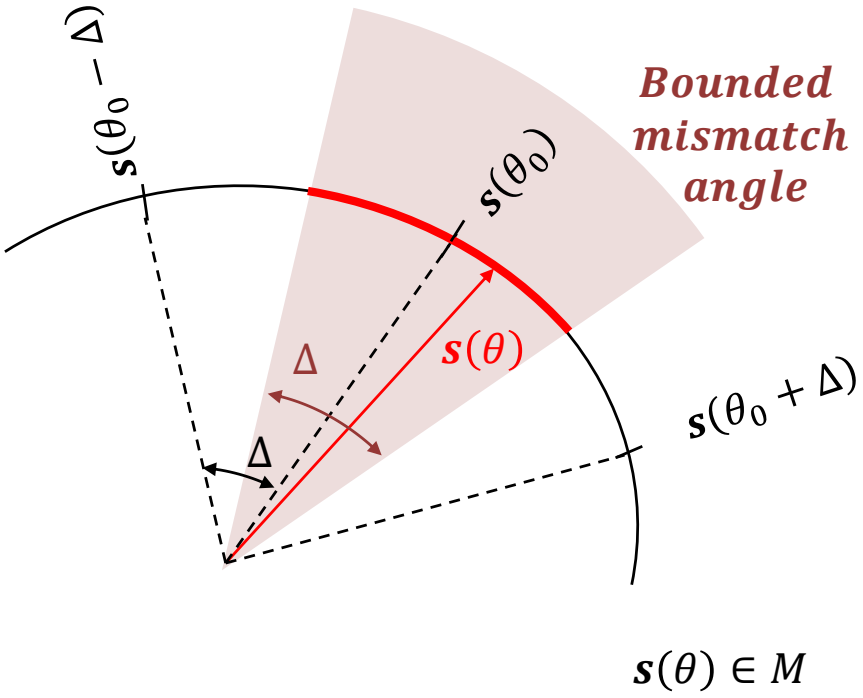
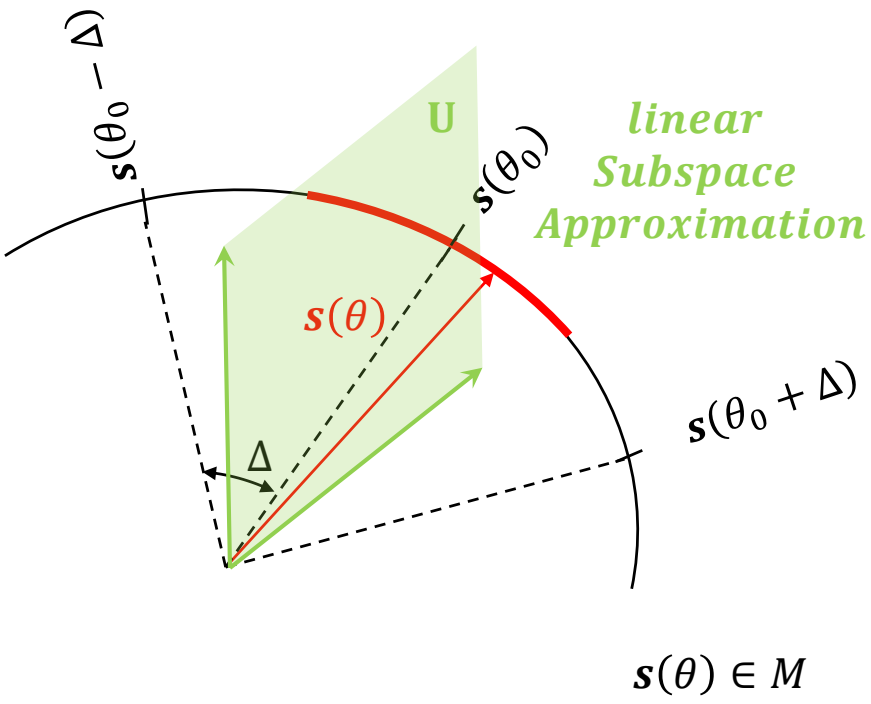


- Choose $\mathbf{A} = \mathbf{s}(\theta_0)$ and cone angle θ_c as the angle between $\mathbf{s}(\theta_0)$ and $\mathbf{s}\left(\theta_0 + \frac{\Delta}{2}\right)$ $(\approx cos^{-1}(\frac{2}{\pi}))$.

The probability for $\mathbf{n} \in \mathscr{C}_{\theta_c}$ sets a limit on the false alarm prob. $(P_{fa}) \Rightarrow$ Not suited for low P_{fa}, moderate P, highly correlated noise.

Common adaptive subspace detectors

- Suited to signals with linear structure: $\mathbf{y} = \mathbf{U}\boldsymbol{\alpha} + \mathbf{n}$ [Kraut et al. 01]
  - ASD: Adaptive Subspace Detector.
  - MAMF: Multirank AMF.
  - CFAR-ASD.
- When $\mathbf{U} = \mathbf{s}(\theta_0)$, ASD = Kelly's test, MAMF=AMF, CFAR-ASD=ACE.
- The tests statistics are, with $\mathbf{z} = \mathbf{S}^{-\frac{1}{2}}\mathbf{y}$, $\mathbf{S} = \frac{1}{K}\sum_{k=1}^{K}\mathbf{y}_k\mathbf{y}_k^H$, $\mathbf{P}_{\mathbf{A}}^{\perp} = \mathbf{I} - \mathbf{P}_{\mathbf{A}}$:



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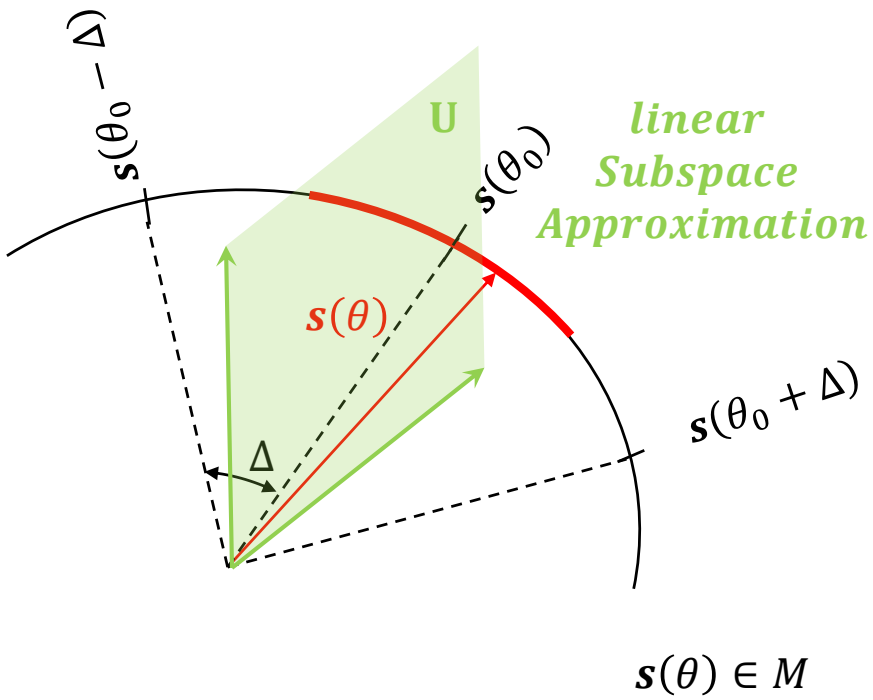

$$t_{ASD} = \frac{\left\|\mathbf{P}_{\mathbf{S}^{-\frac{1}{2}}\mathbf{U}}\mathbf{z}\right\|^2}{K + \left\|\mathbf{P}^{\perp}_{\mathbf{S}^{-\frac{1}{2}}\mathbf{U}}\mathbf{z}\right\|^2}, \quad t_{MAMF} = \frac{\left\|\mathbf{P}_{\mathbf{S}^{-\frac{1}{2}}\mathbf{U}}\mathbf{z}\right\|^2}{K}, \quad t_{CFAR-ASD} = \frac{\left\|\mathbf{P}_{\mathbf{S}^{-\frac{1}{2}}\mathbf{U}}\mathbf{z}\right\|^2}{\left\|\mathbf{P}^{\perp}_{\mathbf{S}^{-\frac{1}{2}}\mathbf{U}}\mathbf{z}\right\|^2},$$

Idea : approximate the set of locally off-grid vectors by a linear subspace

- Analytic performance can be derived

Adaptive subspace detectors performance

- One can sow, using the methodology in [Kraut et al. 01] that

Adaptive subspace statistical distribution

- if $\mathbf{y} = \alpha \mathbf{s}(\theta) + \mathbf{n}$, $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R})$, $\mathbf{U} \in \mathbb{C}^{P \times p}$

$$t_{ASD} = f, \qquad t_{MAMF} = \frac{f}{b}, \qquad t_{CFAR-ASD} = \frac{f}{1-b},$$

  - $f|b \sim F\left(2p, 2(K-P+1), 2b|\alpha|^2 \left\|\mathbf{P}_{\mathbf{R}^{-\frac{1}{2}}\mathbf{U}} \mathbf{R}^{-\frac{1}{2}} \mathbf{s}(\theta)\right\|^2\right)$
  - $1-b \sim \beta\left(2(P-p), 2(K-P+p+1), 2|\alpha|^2 \left\|\mathbf{P}^{\perp}_{\mathbf{R}^{-\frac{1}{2}}\mathbf{U}} \mathbf{R}^{-\frac{1}{2}} \mathbf{s}(\theta)\right\|^2\right)$

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- Test thresholds η_i can be determined for a given $P_{fa} = Pr\left(t_i > \eta_i | \mathcal{H}_0\right)$.
- One is interested in

$$\bar{P}_d^i = \frac{1}{\Delta} \int_{\theta_0 - \frac{\Delta}{2}}^{\theta_0 + \frac{\Delta}{2}} P_d^i\left(\theta\right) d\theta = \frac{1}{\Delta} \int_{\theta_0 - \frac{\Delta}{2}}^{\theta_0 + \frac{\Delta}{2}} Pr\left(t_i(\theta) > \eta_i | \mathcal{H}_1\right) d\theta$$

Subspace choice: the basis $\mathbf{U}$

- Consider for an orthonormal basis the averaged projection residue

$$\mathcal{E}(\theta_0, \Delta, \mathbf{U}) = \frac{1}{\Delta} \int_{\theta_0 - \frac{\Delta}{2}}^{\theta_0 + \frac{\Delta}{2}} \left\| \mathbf{s}(\theta) - \mathbf{U}\mathbf{U}^H \mathbf{s}(\theta) \right\|^2 d\theta.$$

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[Bosse et al. 18][Davenport et al. 12]

The orthonormal matrix $\mathbf{U}$ of dimension p that minimizes $\mathcal{E}(\theta_0, \Delta, \mathbf{U})$ is given by the p strongest eigenvectors of

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We also have $\mathcal{E}(\theta_0, \Delta, \mathbf{U}) = \sum_{k=p+1}^{P} \lambda_k \left(\theta_0, \Delta\right).$

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- In case of sinusoids in noise model :
  - the eigenvectors of $\mathbf{U}(\theta_0, \Delta)$ are deduced from the DPSS vectors [Slepian 78].
  - the eigenvalues $\lambda_k \left(\theta_0, \Delta\right)$ of $\mathbf{U}(\theta_0, \Delta)$ are the DPSS vectors's eigenvalues.

Subspace choice: the dimension p

- Trade-off: when p grows
  - the projection error (the mismatch) decreases :

$$\mathcal{E}\left(\theta_0, \frac{1}{P}, \mathbf{U}\right) = \sum_{k=p+1}^{P} \lambda_k \left(\theta_0, \frac{1}{P}\right).$$

    $\Rightarrow P_d \nearrow$.
  - the threshold increases $\Rightarrow P_d \searrow$.

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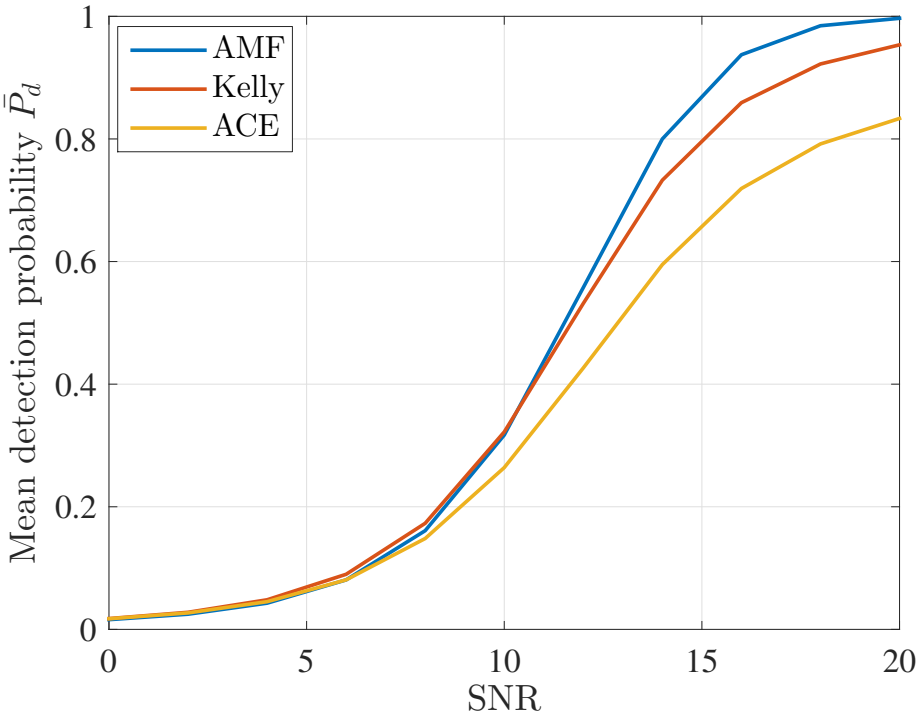
- Why ?
  - When P is large : [Slepian 78] $\lambda_k\left(\theta_0, \frac{1}{P}\right) \approx \lambda_k\left(\frac{\pi}{2}\right)$.
    - $\lambda_k(c)$ is the $k-th$ eigenvalue of the $k-th$ Prolate Spheroidal Wave Function (PSWF) of parameter c.
  - $p = 2$ allows to capture around $98\,\%$ of the signal subspace energy.

Numerical results and conclusion

- $P = 15$,
  $\mathbf{R} = Toeplitz(1, \rho, ..., \rho^{P-1})$,
  $SNR = 10\log_{10}(|\alpha|)$.
- Even in low correlation ($\rho = 0$), the loss can be severe.
- Proposed subspaces offer good performances.



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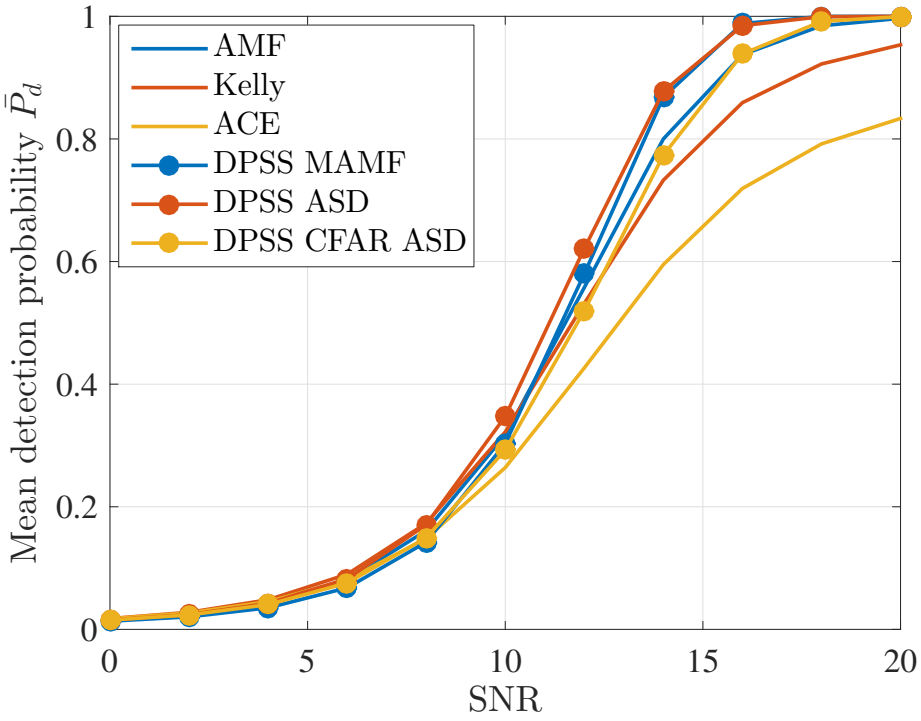

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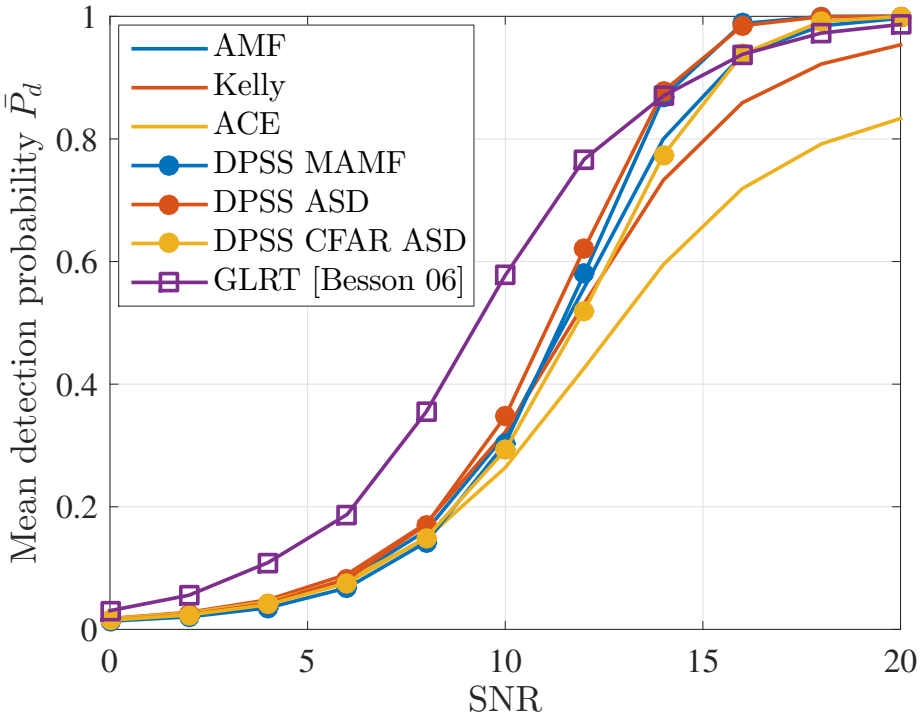

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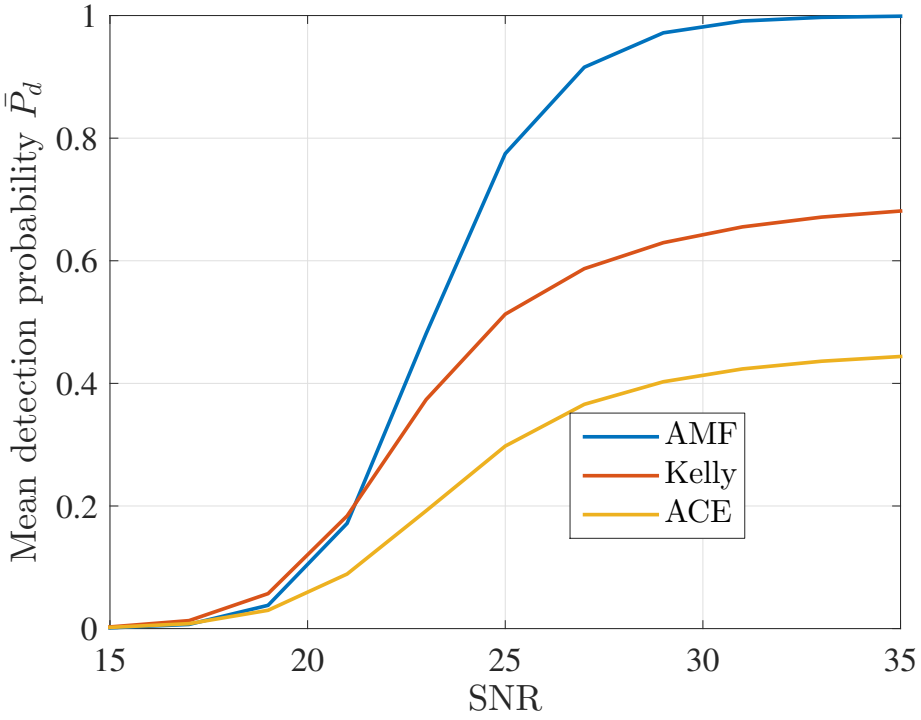

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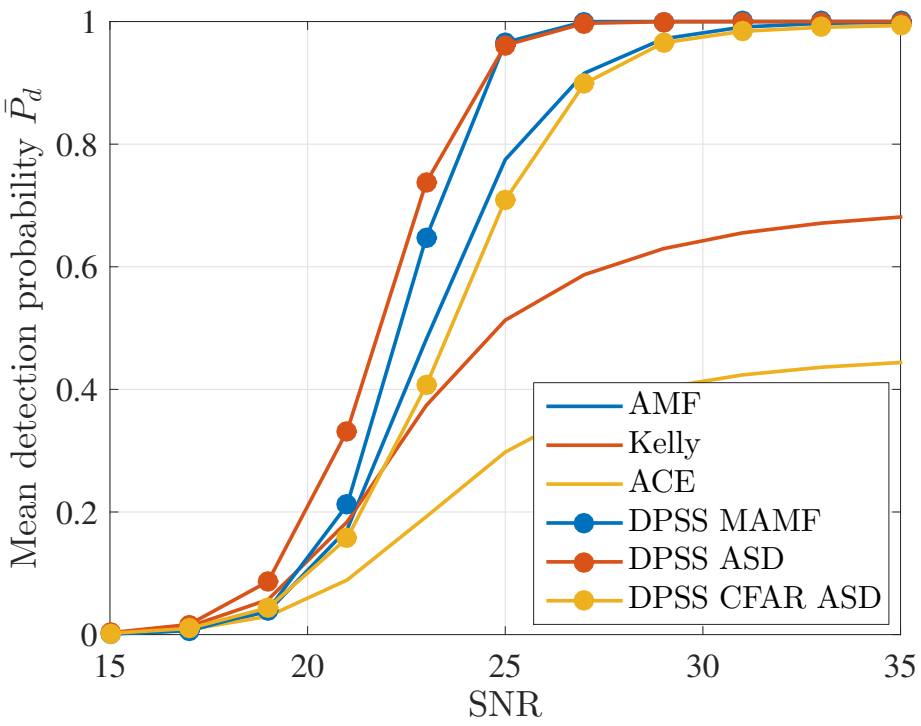

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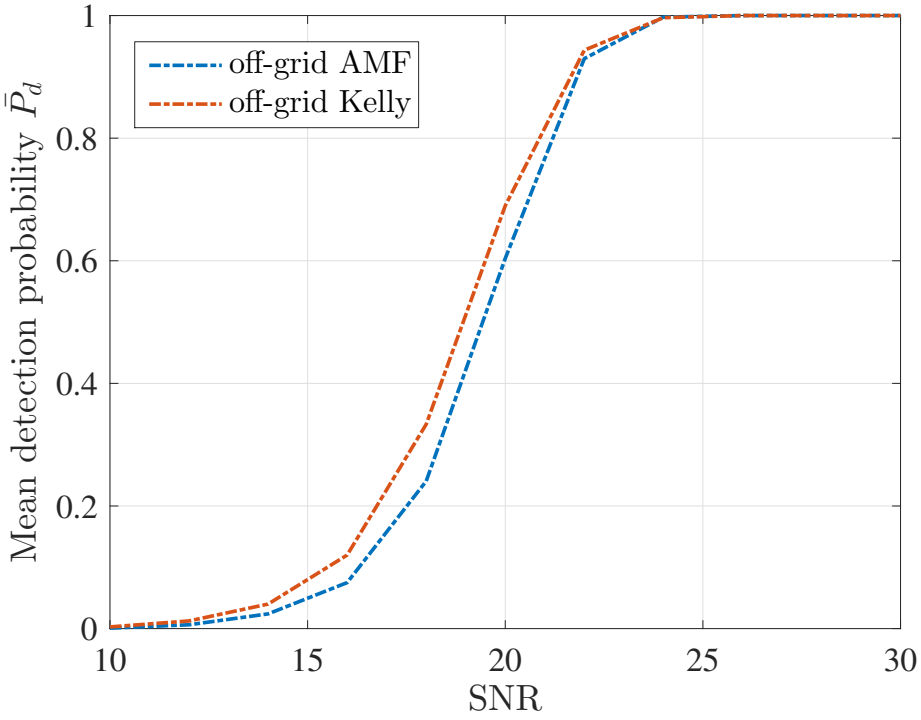

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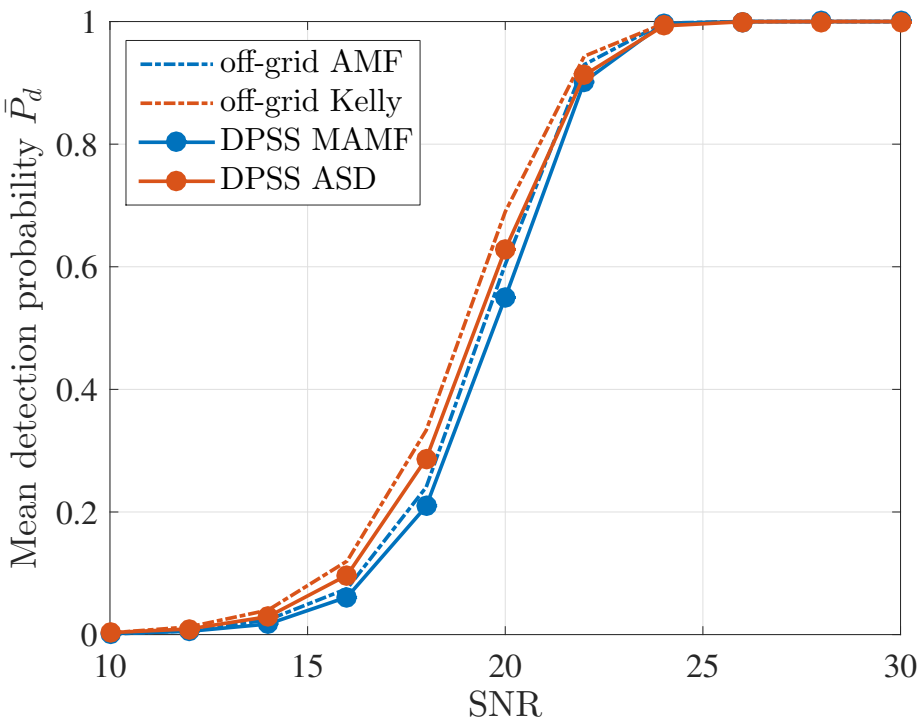

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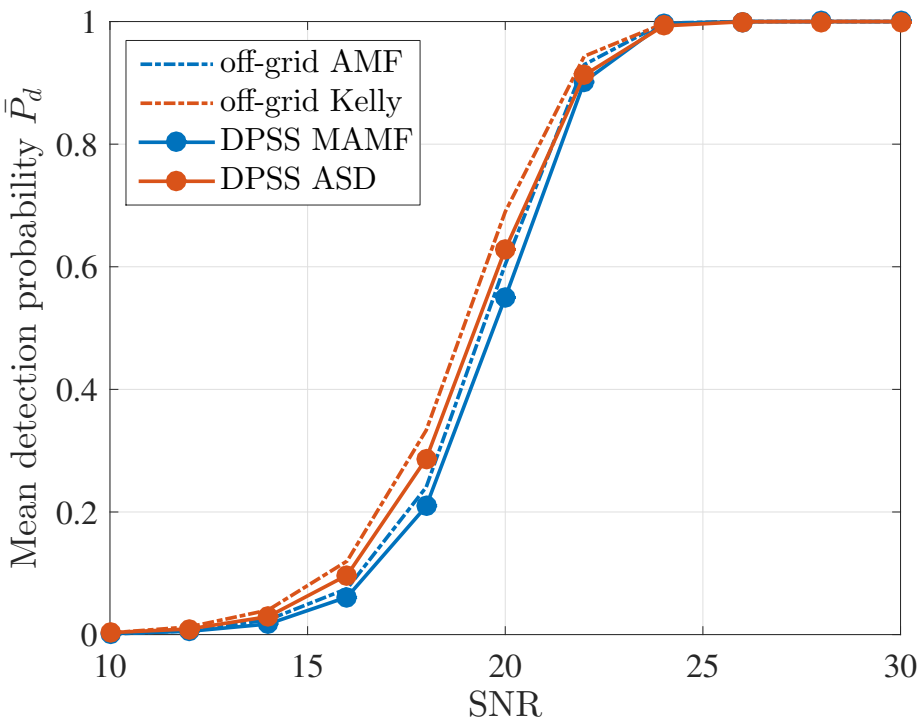
- Optimality of the subspace with respect to the detection probability ?



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Thank you for your attention