Robust Covariance Matrix Estimation and Portfolio Allocation: the case of non-homogeneous assets

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Introduction

- Frequently used portfolio allocation processes require the estimation of the covariance matrix of the asset returns [1, 2, 3, 4]:
  - → The Sample Covariance Matrix (SCM) optimal under the Normal assumption is the most used estimator, but, financial time series might exhibit outliers,
  - → The field of robust estimation intends to deal with outliers [5, 6, 7],
  - → RMT helps in finding a solution for filtering noise [8, 9], but needs to be adapted to non-homogeneous and correlated time series [10].

- In [11] the authors found that considering sub-groups of homogeneous assets may allow for better performance.

- This paper focuses on assets classification methods:
  - → The Affinity Propagation (AP) method [12] that self-determines the number of classes,
  - → The Ascending Hierarchical Clustering (AHC) method that requires the number of classes or determines them using a predefined criterion.

Problem formulation

Variety Maximum (VarMax) portfolio

- The VarMax process, also called the Maximum Diversified Portfolio in [2], allocates assets by maximizing the Variety Ratio ($\mathcal{VR}$) of the portfolio.
- The $\mathcal{VR}$ quantifies the degree of diversification of a portfolio.

VarMax portfolio

Optimal weights $\mathbf{w}^*$ are the weights that maximize the Variety Ratio ($\mathcal{VR}$):

$$\mathbf{w}^* = \underset{\mathbf{w}}{\mathrm{argmax}} \frac{\mathbf{w}^T \mathbf{s}}{(\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w})^{1/2}}, \quad \text{s.t. } \mathbf{w}^T \mathbf{1}_m = 1 \text{ and } 0 \leq w_i \leq 1, \forall i \in [1, m],$$

where $\boldsymbol{\Sigma}$ is the covariance matrix and $\mathbf{s}$ such that $s_i = \sqrt{\boldsymbol{\Sigma}_{ii}}$, $i \in [1, m]$.

$\Rightarrow$ We focus on the VarMax process because it is the allocation process most sensitive to the covariance matrix of the assets returns.

$\Rightarrow$ Problem: $\boldsymbol{\Sigma}$ unknown $\rightarrow$ need to be estimated

Let $\mathbf{R} = \{\mathbf{r}_t\}_{t\in[1,N]}$ be the $m \times N$-matrix containing the N observations of the m asset returns, modelled as a K-factor model [13, 14] with an additive multivariate Elliptic Symmetric distributed noise [15, 16]. For each observation date $t \in [1, N]$, we then have:

$$\mathbf{r}_t = \mathbf{B}_t\, \mathbf{f}_t + \sqrt{\tau_t}\, \mathbf{C}^{1/2}\, \mathbf{x}_t, \tag{1}$$

- $\mathbf{r}_t$ is the m-vector of returns,
- $\mathbf{B}_t$ is the $m \times K$-matrix of coefficients that define the assets sensitivities of the K factors,
- $\mathbf{f}_t$ is the K-vector of random factors and common to the m assets,
- $\mathbf{x}_t$ is the m-vector of independent Gaussian white noise with unit variance and is non-correlated with the factors,
- $\mathbf{C}$ is called the $m \times m$ scatter matrix that is Toeplitz structured [17] and is time invariant over the period of observation,
- τ_t is an i.i.d positive random variables with expectation τ that is independent of the noise and the factors and drives the variance of the noise.

Problem formulation

Whitening process

Robust Consistent Estimation for **C** [18]

Let $\hat{\mathbf{C}}_{tyl} = \dfrac{m}{N} \displaystyle\sum_{t=1}^{N} \dfrac{\mathbf{r}_t\, \mathbf{r}_t^T}{\mathbf{r}_t^T\, \hat{\mathbf{C}}_{tyl}^{-1}\, \mathbf{r}_t}$ be the scatter matrix Tyler M-estimator of **R**.

As $m, N \to \infty$ such that $m/N \to c \in]0, \infty[$, we have $\left\| \mathcal{T}\left[\hat{\mathbf{C}}_{tyl}\right] - \mathbf{C} \right\| \xrightarrow{a.s.} 0$,

where $\mathcal{T}[\cdot]$ is the **Toeplitz rectification** operator $(\mathcal{T}[\mathbf{A}])_{ij} = \dfrac{1}{m} \displaystyle\sum_{l=i}^{m} a_{l,l-i+1}$.

A consistent estimator $\widetilde{\mathbf{C}}_{tyl}$ of the background scatter matrix **C** is therefore defined through observations **R** as $\widetilde{\mathbf{C}}_{tyl} = \mathcal{T}\left[\hat{\mathbf{C}}_{tyl}\right]$.

$\Longrightarrow$ The observations **R** can now be whitened through $\widetilde{\mathbf{C}}_{tyl}^{-1/2}\,\mathbf{R}$

Problem formulation

Whitening process

Behavior of whitened data [18]

Let $\mathbf{R}_w = \left(\mathcal{T}\left[\hat{\mathbf{C}}_{tyl}\right]\right)^{-1/2} \mathbf{R}$ be the whitened data and $\hat{\boldsymbol{\Sigma}}_{tyl}$ be the Tyler M-estimator of $\mathbf{R}_w$. As $m, N \to \infty$ such that $m/N \to c \in]0, \infty[$, if $\mathbf{R}_w$ does not contain any factor, then:

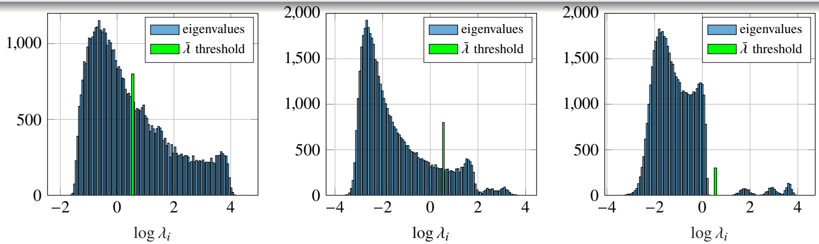
$$\left\| \hat{\boldsymbol{\Sigma}}_{tyl} - \frac{1}{N} \mathbf{X}\mathbf{X}^T \right\| \xrightarrow{a.s.} 0\,.$$

- Without factors, the spectral distribution of the whitened data scatter matrix of $\mathbf{R}_w$ follows a Marchenko-Pastur distribution [19, 20] (same spectral distribution of unobservable covariance matrix of $\mathbf{X}$) characterized by its support $\left[\left(1-\sqrt{c}\right)^2, \left(1+\sqrt{c}\right)^2\right]$,
- All eigenvalues greater than $\bar{\lambda} = \left(1+\sqrt{c}\right)^2$ can be considered as significant factors.

Estimation of K the number of factors

Let $(\lambda_k)_{k\in[1,m]}$ be the sorted eigenvalues of $\hat{\mathbf{\Sigma}}_{tyl}$, then:

$$\hat{K} = \underset{k}{\operatorname{argmax}} \left(\lambda_k > \bar{\lambda}\right)$$



Eigenvalue distributions [10]. Left: $\mathbf{R}\,\mathbf{R}^T/N$, Sample Covariance Matrix of observations. Middle: $\hat{\mathbf{C}}_{tyl}$, Tyler covariance matrix of observations. Right: $\hat{\mathbf{\Sigma}}_{tyl}$, Tyler covariance matrix of observations after whitening process. K-distributed case with shape parameter $\nu = 0.5$, $\rho = 0.8$, $m = 100$, $N = 1000$, $K = 3$.

Problem formulation

Non-homogeneous asset returns model

- The model (1) and the whitening process described above is made under the implicit assumption that the asset returns are drawn from a unique multivariate law but this assumption is unrealistic for financial time series of returns.
- To take into account the non-homogeneous asset returns, the model (1) is rewritten for the m assets splitted into $p < m$ groups. Each group is composed of $\{m_q\}_{q=1}^{p}$ assets, and composed of assets with similar distributions. It follows that:

$$\mathbf{r}_t^{(q)} = \mathbf{B}_t^{(q)} \mathbf{f}_t + \sqrt{\tau_t^{(q)}} \, \mathbf{C}_{(q)}^{1/2} \mathbf{x}_t, \tag{2}$$

$\Rightarrow$ The complete scatter matrix $\mathbf{C}$ is therefore block-constructed, block-Toeplitz, and the groups are assumed to be uncorrelated to each other.

- Under the assumption of non-homogeneous asset returns, we propose to form groups of assets before applying the whitening process.

- Two clustering methods are compared to form the groups of assets:

  ⇒ The Affinity Propagation algorithm (AP) [12] that does not require to specify the number of groups,

  ⇒ The classical Ascending Hierarchical Classification (AHC), where the number of groups p is determined arbitrarily or with Caliński-Harabasz (CH) criterion [21].

The Affinity Propagation algorithm (AP) [12]:

- an iterative partitioning method similar to the K-means, but it regroups individuals around exemplar values,
- is based on a similarity matrix $\mathbf{S}$, where $s_{i,j} = -\|\mathbf{v}_i - \mathbf{v}_j\|^2$ for $i \neq j$, and with $\mathbf{v}_i$ and $\mathbf{v}_j$ the input variables vectors of the asset i and j,
- to moderate, the number of groups p, the parameters are set to a common value using the median of pairwise similarities as in [12].

The classical Ascending Hierarchical Classification (AHC):

- is an iterative and unsupervised method,
- is based on the distances between the variables $(\mathbf{v}_i)_{i\in[1,m]}$ used to represent individuals to be grouped and seeks at each step to build the groups by aggregation,
- is used with the Euclidean distance and the Ward measure [22] to form the p groups.

Given $\mathbf{R}$ the $m \times N$-matrix of observations, and $\mathbf{R}^{(q)}$ the $m_q \times N$-matrix of observations for group (q), the whitened asset returns $\mathbf{R}_w$ are obtained through the following procedure:

- Compute the p groups using the methods described previously with $(\mathbf{v}_i)_{i\in[1,m]}$ composed of the mean μ_i, the standard deviation σ_i and of several quantiles computed from $\tilde{\mathbf{r}}_i = (\mathbf{r}_i - \mu_i \, \mathbf{1}_N) / \sigma_i$ the "standardized" returns, where $\mathbf{1}_N$ is the N-vector of ones,
- Set $\widehat{\mathbf{C}}_{tyl}^{(q)}$ the Tyler-M estimate of $\mathbf{R}^{(q)}$,
- Set $\widetilde{\mathbf{C}}_{tyl}^{(q)} = \mathcal{T}\left(\widehat{\mathbf{C}}_{tyl}^{(q)}\right)$, the Toeplitz rectification matrix built from $\widehat{\mathbf{C}}_{tyl}^{(q)}$ for the Toeplitz operator $\mathcal{T}$,
- Set $\mathbf{R}_w^{(q)} = \left(\widetilde{\mathbf{C}}_{tyl}^{(q)}\right)^{-1/2} \mathbf{R}^{(q)}$, the $m_q \times N$ matrix of the whitened observations of group q,

Proposed methodology

Detailed whitening procedure

Finally, the de-noised covariance matrix estimate $\widehat{\boldsymbol{\Sigma}}_w$ is obtained as follows:

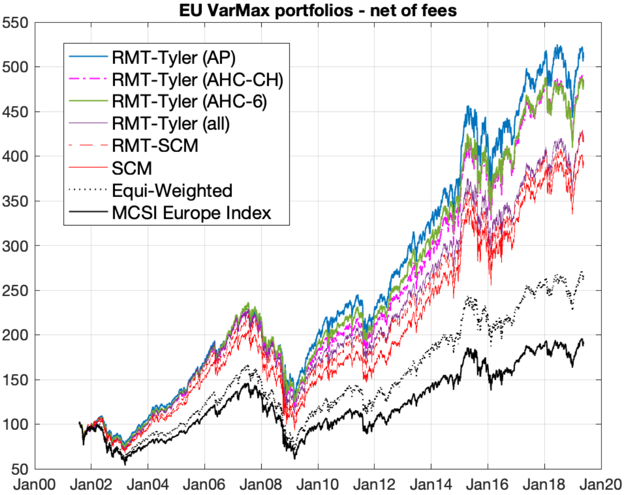
- Set $\widehat{\boldsymbol{\Sigma}}_{tyl}$ as the Tyler-M estimate of $\mathbf{R}_w$, where $\mathbf{R}_w = \left[\mathbf{R}_w^{(1)\,T} \ldots \mathbf{R}_w^{(p)\,T}\right]^T$ of size $m \times N$,
- Set $\widehat{\boldsymbol{\Sigma}}_{tyl}^{clip} = \mathbf{U}\,\boldsymbol{\Lambda}^{clip}\,\mathbf{U}^T$ where $\mathbf{U}$ is the $m \times m$ eigenvectors matrix and $\boldsymbol{\Lambda}^{clip}$ is the $m \times m$ diagonal matrix of the eigenvalues $(\lambda_k^{clip})_{k\in[1,m]}$ corrected using the Eigenvalue clipping method [23]:

$$\lambda_k^{clip} = \begin{cases} \lambda_k, & \text{if } \lambda_k \geq \left(1+\sqrt{c}\right)^2 \\ \dfrac{1}{m-K}\left(\displaystyle\sum_{k=1}^{m}\lambda_k - \sum_{k=1}^{K}\lambda_k\right), & \text{otherwise} \end{cases}$$

- Finally, $\widehat{\boldsymbol{\Sigma}}_w = \left(\widetilde{\mathbf{C}}_{tyl}^{1/2}\right)\widehat{\boldsymbol{\Sigma}}_{tyl}^{clip}\left(\widetilde{\mathbf{C}}_{tyl}^{1/2}\right)^T$.

- Two investment universes are tested:
  - $\Rightarrow$ European equity indices ($m = 43$): countries, sub-sectors and factors.
  - $\Rightarrow$ US equity indices ($m = 30$): sub-sectors and factors.
- Optimization settings:
  - $\Rightarrow$ daily closing prices from July 27th, 2000 to May 20th, 2019,
  - $\Rightarrow$ the covariance matrix of the assets is estimated using the past daily returns ($N = 260$),
  - $\Rightarrow$ the portfolio weights are computed every four weeks and kept constant for the next four-weeks period.
- Clustering method settings:
  - $\Rightarrow$ the quantiles used are q_θ and $q_{1-\theta}$ with $\theta \in \{1\%, 2.5\%, 5\%, 10\%, 15\%, 25\%, 50\%\}$,
  - $\Rightarrow$ for AHC method, $p = 6$ ("AHC-6") as in [11] or set according to the CH criterion ("AHC-CH").
- The portfolio performances are net of transaction fees (0.07%) to take into account the portfolio turnover.

Application
EU portfolio results

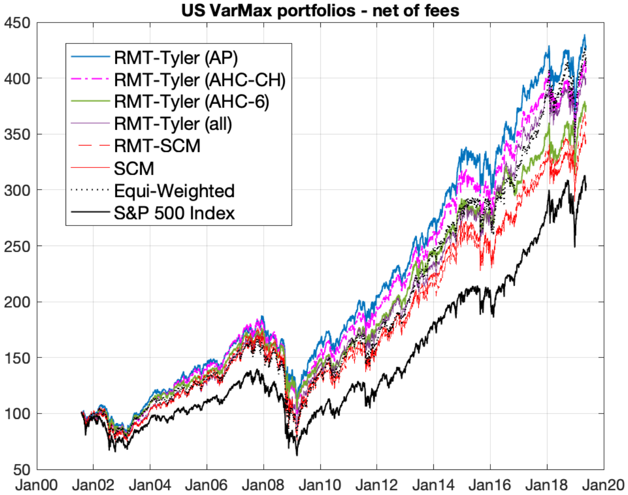


EU VarMax portfolios' wealth with 0.07% of fees from July 2001 to May 2019.

EU VarMax Portfolios	Ann. Ret.	Ann. Vol.	Ratio Ret/Vol	Max Drawdown	$\mathcal{VR}$ (avg)
RMT-Tyler (AP)	9.87%	12.14%	0.81	45.37%	1.46
RMT-Tyler (AHC-6)	9.65%	12.03%	0.80	46.84%	1.57
RMT-Tyler (AHC-CH)	9.58%	12.45%	0.77	48.16%	1.51
RMT-Tyler (all)	8.90%	13.16%	0.68	51.18%	1.44
RMT-SCM	8.94%	13.79%	0.65	54.15%	1.27
SCM	8.56%	13.68%	0.63	54.45%	1.38
Equi-Weighted	6.60%	15.37%	0.43	57.82%	1.19
MSCI Europe	*4.71%*	*14.87%*	*0.32*	*58.54%*	

Performance numbers for the Europe (EU) VarMax portfolios with 0.07% of fees from July 2001 to May 2019.

Application
US portfolio results



US VarMax portfolios' wealth with 0.07% of fees from July 2001 to May 2019.

Application

US portfolio results

US VarMax Portfolios	Ann. Ret.	Ann. Vol.	Ratio Ret/Vol	Max Drawdown	$\mathcal{VR}$ (avg)
RMT-Tyler (AP)	8.76%	11.11%	0.79	42.82%	1.51
RMT-Tyler (AHC-CH)	8.57%	11.53%	0.74	46.57%	1.55
RMT-Tyler (AHC-6)	7.98%	10.79%	0.74	41.50%	1.52
RMT-Tyler (all)	8.49%	12.09%	0.70	49.27%	1.53
Equi-Weighted	8.92%	13.83%	0.65	53.70%	1.25
RMT-SCM	8.03%	13.13%	0.61	56.53%	1.34
SCM	7.80%	13.27%	0.59	55.47%	1.46
S&P 500	*7.21%*	*14.18%*	*0.51*	*55.71%*	

Performance numbers for the US VarMax portfolios with 0.07% of fees from July 2001 to May 2019.

Conclusion

- Asset returns have been modelled as a multi-factor model embedded in a correlated elliptical and symmetric noise by considering that the asset returns are non-homogeneous in law which is more realistic,
- Given this model setup, we question the ability of classification methods (AP algorithm and AHC) to improve whitening process based on the Tyler M-estimator and the RMT,
- Our methodology has been tested on the Maximum Variety portfolio optimization problem and proves the superiority of the AP algorithm in producing higher performances for both EU and US universes.

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