NEW INSIGHT ON TYLER'S SHAPE MATRIX BASED ESTIMATORS AND PERSYMMETRIC STRUCTURE

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ABSTRACT

This paper deals with persymmetric structured shape matrix estimation under compound-Gaussian distributions. In the framework of robust estimation, the compound-Gaussian distributions are particularly relevant to describe heterogeneous environment. In this context, we emphasize the connections between persymmetric structured shape matrix estimators based on Tyler's estimate but derived from different methodologies, notably the extended invariance principle or the Euclidean projection.

Index Terms— Persymmetric matrix, shape matrix, Tyler estimator, structured estimation.

1. INTRODUCTION

Adaptive radar detection in a heterogeneous environment remains an important challenge in the age of high-resolution radars. Most of the existing algorithms in the literature require the estimation of the Clutter Covariance Matrix (CCM) from secondary data [1, 2]. For several years already, the heterogeneity of the environment is modeled with non-Gaussian processes, notably by Compound-Gaussian (CG) distributed random vectors [3,4]. Some on-field measurements have shown that CG distribution could accurately model spiky radar clutter [5, 6]. Furthermore, the CCM may exhibit a particular structure in addition to its Hermitian symmetry and positive definiteness. For example a linear array, which is symmetrically spaced with respect to (w.r.t.) the phase center leads to the persymmetric structure [7]. Taking into account this particular structure of the CCM in the estimation scheme provides lower requirements on the secondary data. Moreover, that leads to a better estimation accuracy, since the degree of freedom in the estimation problem decreases [8]. Several robust methods, based on the Tyler's estimate [9], have been proposed to address the problem of persymmetric structured CCM with CG distributed clutter [10–14]. In this context, the main contribution of this paper is to highlight the links between different estimators based on distinct methodologies, especially those of [11] and [14].

This paper is organized as follows. In section 2, we bind our contribution to prior work. In section 3, a brief review on the Hermitian persymmetric matrices and the Compound-Gaussian distribution is presented. Section 4 focuses on persymmetric structured shape matrix estimators and the connections between the different approaches.

In the following, the notation $\stackrel{d}{=}$ indicates "has the same distribution as". For a matrix $\text{Tr}\,(\mathbf{A})$ denotes the trace of $\mathbf{A}$. $\mathbf{A}^T$ (respectively $\mathbf{A}^H$ and $\mathbf{A}^*$) stands for the transpose (respectively conjugate transpose and conjugate) matrix. The vec-operator vec$(\mathbf{A})$ stacks all columns of $\mathbf{A}$ into a vector. The identity matrix of size m is referred to $\mathbf{I}_m$. The imaginary unit is denoted by j. The operator $\otimes$ refers to Kronecker matrix product and finally, the subscript "e" refers to the true value.

2. RELATION TO PRIOR WORK

In order to take into account the persymmetric structure of the CCM, a unitary transformation applied to the Tyler's estimate is used in [11], which can be interpreted as an Euclidean projection over the subset of Hermitian persymmetric matrices. In [13], the authors present robust covariance estimators under group symmetry constraints by creating synthetic data from the existing ones with appropriate permutations. However, our definition of usual Hermitian persymmetric matrices does not belong to this framework. A robust extension of the Covariance Matching Estimation Technique (COMET) is derived in [14] for convex structured matrices. This approach is originally based on the extended invariance principle [15]. In this paper, we do not attempt to derive a new estimator, but we rather highlight the links between two existing estimators, presented in [11] and [14], based on different approaches for a better understanding of Hermitian persymmetric structured matrices estimation. Theses methods, though independently derived, are surprisingly related.

3. BACKGROUND AND PROBLEM SETUP

3.1. Persymmetric matrix

Let $\mathbf{L}_m \in \mathbb{R}^{m\times m}$ be the m-dimensional antidiagonal matrix, having 1 as non-zero element and the unitary matrix, $\mathbf{T} \in \mathbb{C}^{m\times m}$, defined as:

$$\mathbf{T} = \begin{cases} \dfrac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{I}_{m/2} & \mathbf{L}_{m/2} \\ j\mathbf{I}_{m/2} & -j\mathbf{L}_{m/2} \end{pmatrix} & \text{for } m \text{ even} \\ \dfrac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{I}_{(m-1)/2} & 0 & \mathbf{L}_{(m-1)/2} \\ 0 & \sqrt{2} & 0 \\ j\mathbf{I}_{(m-1)/2} & 0 & -j\mathbf{L}_{(m-1)/2} \end{pmatrix} & \text{for } m \text{ odd} \end{cases}$$

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and verifying the relation $\mathbf{T}\mathbf{T}^T = \mathbf{L}_m$. A Hermitian matrix, $\mathbf{A} \in \mathbb{C}^{m\times m}$, has the persymmetric property if it satisfies one of the following equivalent conditions:

(i) $\mathbf{A} = \mathbf{L}_m\mathbf{A}^*\mathbf{L}_m = \mathbf{L}_m\mathbf{A}^T\mathbf{L}_m$

(ii) $\mathbf{T}\mathbf{A}\mathbf{T}^H$ is a real symmetric matrix.

3.2. Compound-Gaussian distribution

A m-dimensional zero mean random vector (r.v.), $\mathbf{y} \in \mathbb{C}^m$ has a Complex Compound-Gaussian distribution, denoted by $\mathbf{y} \sim \mathbb{C}\mathcal{CN}_m(\mathbf{0}_m, \mathbf{M}, p_\tau)$ if it admits the following stochastic representation:

$$\mathbf{y} \stackrel{d}{=} \sqrt{\tau}\mathbf{g} \tag{1}$$

where τ is a non-negative scalar random variable, having the density of probability p_τ and the r.v. $\mathbf{g}$ is circular complex Gaussian distributed with zero-mean, $\mathbf{g} \sim \mathbb{C}\mathcal{N}_m(\mathbf{0}_m, \mathbf{M})$. The variables τ, called the *texture*, and $\mathbf{g}$, referred to as the *speckle*, are independent. For identifiability consideration, a constraint should be added on the texture or the covariance matrix of the speckle: in the following, we normalize the matrix $\mathbf{M}$ such as $\mathrm{Tr}(\mathbf{M}) = m$. With respect to the r.v. $\mathbf{y}$, the matrix $\mathbf{M}$ is referred to the shape matrix.
From a set of N i.i.d. CG distributed data, $\mathbf{y}_n \sim \mathbb{C}\mathcal{CN}_m(\mathbf{0}_m, \mathbf{M}, p_\tau)$, $n = 1, \dots, N$ with $N > m$, the well-known Tyler's estimate is a robust estimator of the shape matrix, which is the solution of the following fixed-point equation [2, 9]:

$$\widehat{\mathbf{M}}_{\text{FP}} = \frac{m}{N}\sum_{n=1}^{N}\frac{\mathbf{y}_n\mathbf{y}_n^H}{\mathbf{y}_n^H\widehat{\mathbf{M}}_{\text{FP}}^{-1}\mathbf{y}_n} \triangleq \mathcal{H}\left(\widehat{\mathbf{M}}_{\text{FP}}\right) \tag{2}$$

Existence and uniqueness up to a scale factor of the solution of the above equation have been studied in [16]. In the following, we apply a constraint on the trace, $\mathrm{Tr}\left[\widehat{\mathbf{M}}_{\text{FP}}\right] = m$, to avoid ambiguity of the solution of (2). The solution $\widehat{\mathbf{M}}_{\text{FP}}$ is obtained by an iterative algorithm, $\mathbf{M}_{k+1} = \mathcal{H}(\mathbf{M}_k)$ with the normalization on the trace, which converges to $\widehat{\mathbf{M}}_{\text{FP}}$, for any initialization point [9, 17]. Furthermore, $\widehat{\mathbf{M}}_{\text{FP}}$ is a consistent and unbiased estimator of $\mathbf{M}$ [17].

Remark: In the general case, with τ a non-negative random variable, the Tyler's estimate is an approximate Maximum Likelihood (ML) of the shape matrix. If we relax τ to an unknown deterministic parameter, the Tyler's estimate coincides with the ML estimator of the shape matrix.

3.3. Problem setup

Let us consider N i.i.d. zero mean CG distributed observations, $\mathbf{y}_n \sim \mathbb{C}\mathcal{CN}_m(\mathbf{0}_m, \mathbf{M}_e, p_\tau)$, $n = 1, \dots, N$ with $N > m$. We assume that the shape matrix belongs to a convex subset $\mathscr{S}$ of Hermitian positive-definite matrices with a persymmetric structure. Consequently, there exists a one-to-one differentiable mapping $\boldsymbol{\mu} \mapsto \boldsymbol{\mathcal{M}}(\boldsymbol{\mu})$ from $\mathbb{R}^p$ to $\mathscr{S}$, with p detailed later. The vector $\boldsymbol{\mu}$ is the unknown parameter of interest, with exact value $\boldsymbol{\mu}_e$ and $\mathbf{M}_e = \boldsymbol{\mathcal{M}}(\boldsymbol{\mu}_e)$. In the particular case of the persymmetric structure, a natural parameterization is as follows, for m even:

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\mu}) = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2^* & a_5 & a_6 & a_3 \\ a_3^* & a_6^* & a_5 & a_2 \\ a_4^* & a_3^* & a_2^* & a_1 \end{pmatrix} \text{ and } \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \vdots \\ \boldsymbol{\mu}_\ell \end{pmatrix} \in \mathbb{R}^p$$

with $\ell = \dfrac{m(m+2)}{4}$ and for $k = 1, \dots, \ell$

$$\boldsymbol{\mu}_k = \begin{cases} a_k & \text{if } a_k \in \mathbb{R} \\ \begin{pmatrix} \Re(a_k) \\ \Im(a_k) \end{pmatrix} & \text{otherwise} \end{cases}$$

The case m odd is quite similar with $\ell = \dfrac{(m+1)^2}{4}$. Furthermore, the length of the vector $\boldsymbol{\mu}$ is always equal to $p = \dfrac{m(m+1)}{2}$. Hence, there exists a full column rank matrix $\boldsymbol{\mathcal{J}} \in \mathbb{C}^{m^2\times p}$, which relates the vectorized matrix $\boldsymbol{\mathcal{M}}(\boldsymbol{\mu})$ to $\boldsymbol{\mu}$ as

$$\boldsymbol{\eta}(\boldsymbol{\mu}) = \mathrm{vec}(\boldsymbol{\mathcal{M}}(\boldsymbol{\mu})) = \boldsymbol{\mathcal{J}}\boldsymbol{\mu} \tag{3}$$

The full column rank matrix $\boldsymbol{\mathcal{J}}$ admits a left inverse $\boldsymbol{\mathcal{J}}^\dagger = \left(\boldsymbol{\mathcal{J}}^H\boldsymbol{\mathcal{J}}\right)^{-1}\boldsymbol{\mathcal{J}}^H$ verifying $\boldsymbol{\mathcal{J}}^\dagger\boldsymbol{\mathcal{J}} = \mathbf{I}_p$ [18].

In this paper, we highlight the connections between several structured shape matrix estimators based on different approaches.

4. PERSYMMETRIC STRUCTURED SHAPE MATRIX ESTIMATORS

In this section, we recall two different estimators of a persymmetric structured shape matrix. We demonstrate that they lead to the same result although derivated from different approaches.

4.1. RCOMET based estimator

The first studied structured shape matrix estimator is based on a recent robust extension of COMET method presented in [14]: the Robust COMET (RCOMET) estimate which is obtained by

$$\begin{aligned}\widehat{\boldsymbol{\mu}}_0 &= \arg\min_{\alpha,\boldsymbol{\mu}} \mathrm{Tr}\left[\left\{\left(\widehat{\mathbf{M}}_{\text{FP}} - \alpha\boldsymbol{\mathcal{M}}(\boldsymbol{\mu})\right)\widehat{\mathbf{M}}_{\text{FP}}^{-1}\right\}^2\right] \text{ s.t. } \mathrm{Tr}[\boldsymbol{\mathcal{M}}(\boldsymbol{\mu})] = m \\ &= \arg\min_{\alpha,\boldsymbol{\mu}} \left\|\widehat{\mathbf{W}}_{\text{FP}}^{-1/2}(\widehat{\boldsymbol{\eta}}_{\text{FP}} - \alpha\boldsymbol{\mathcal{J}}\boldsymbol{\mu})\right\|^2 \text{ s.t. } \mathrm{Tr}[\boldsymbol{\mathcal{M}}(\boldsymbol{\mu})] = m \end{aligned} \tag{4}$$

where $\widehat{\boldsymbol{\eta}}_{\text{FP}} = \mathrm{vec}\left(\widehat{\mathbf{M}}_{\text{FP}}\right)$ and $\widehat{\mathbf{W}}_{\text{FP}} = \widehat{\mathbf{M}}_{\text{FP}}^T \otimes \widehat{\mathbf{M}}_{\text{FP}}$. The coefficient $\alpha > 0$ is required for the purpose of theoretical derivation, since the constraints on the Tyler's estimate and the parameterized shape matrix could be different (cf [14] for more details). The minimization of (4) w.r.t $\alpha\boldsymbol{\mathcal{M}}$ over $\mathscr{S}$ is a convex problem that admits a unique solution. Finally, the one-to-one mapping and the constraint on the trace yield a unique solution for $\boldsymbol{\mu}$. In addition, we can easily see that $\widehat{\boldsymbol{\mu}}_0 \propto \widehat{\boldsymbol{\lambda}}$, with

$$\begin{aligned}\widehat{\boldsymbol{\lambda}} &= \arg\min_{\boldsymbol{\lambda}} \left\|\widehat{\mathbf{W}}_{\text{FP}}^{-1/2}(\widehat{\boldsymbol{\eta}}_{\text{FP}} - \boldsymbol{\mathcal{J}}\boldsymbol{\lambda})\right\|^2 \\ &= \left(\boldsymbol{\mathcal{J}}^H\widehat{\mathbf{W}}_{\text{FP}}^{-1}\boldsymbol{\mathcal{J}}\right)^{-1}\boldsymbol{\mathcal{J}}^H\widehat{\mathbf{W}}_{\text{FP}}^{-1}\widehat{\boldsymbol{\eta}}_{\text{FP}}\end{aligned}$$

The trace constraint leads to

$$\widehat{\boldsymbol{\mu}}_0 = \frac{m}{\mathrm{vec}(\mathbf{I}_m)^T\boldsymbol{\mathcal{J}}\widehat{\boldsymbol{\lambda}}}\widehat{\boldsymbol{\lambda}} \tag{5}$$

The estimate $\widehat{\boldsymbol{\mu}}_0$ (respectively $\boldsymbol{\mathcal{M}}(\widehat{\boldsymbol{\mu}}_0)$) is consistent to $\boldsymbol{\mu}_e$ (respectively $\mathbf{M}_e$). A recursive implementation of RCOMET procedure, referred as Recursive RCOMET (R-RCOMET), yields the same asymptotic performance of RCOMET and can be achieved at the k-th iteration by

$$\widehat{\boldsymbol{\mu}}_k = \arg\min_{\alpha,\boldsymbol{\mu}} \mathrm{Tr}\left[\left\{\left(\widehat{\mathbf{M}}_{\mathrm{FP}} - \alpha\boldsymbol{\mathcal{M}}\left(\boldsymbol{\mu}\right)\right)\boldsymbol{\mathcal{M}}\left(\widehat{\boldsymbol{\mu}}_{k-1}\right)^{-1}\right\}^2\right] \quad \text{s.t. } \mathrm{Tr}\left[\boldsymbol{\mathcal{M}}\left(\boldsymbol{\mu}\right)\right] = m$$

with $\widehat{\boldsymbol{\mu}}_0$ given by (5). The R-RCOMET estimate is the estimate at the stage $K < \infty$, denoted by $\widehat{\boldsymbol{\mu}}_K$.

4.2. Projected Tyler estimate

The second studied structured shape matrix estimator, presented in [11], uses the unitary matrix $\mathbf{T}$ and the condition (ii) for the persymmetric structure. The structured shape matrix estimate, denoted by Persymmetric Fixed-Point (PFP) estimate, is obtained by

$$\widehat{\mathbf{M}}_{\mathrm{PFP}} = \mathbf{T}^H \Re\left(\mathbf{T}\widehat{\mathbf{M}}_{\mathrm{FP}}\mathbf{T}^H\right)\mathbf{T} \qquad (6)$$

where $\widehat{\mathbf{M}}_{\mathrm{FP}}$ is the Tyler's estimate defined in (2). By vectorizing (6), we can define $\widehat{\boldsymbol{\mu}}_{\mathrm{PFP}}$ given by

$$\begin{aligned}
\widehat{\boldsymbol{\mu}}_{\mathrm{PFP}} &= \boldsymbol{\mathcal{J}}^{\dagger}\left(\mathbf{T}^T \otimes \mathbf{T}^H\right)\mathrm{vec}\left(\frac{1}{2}\left[\mathbf{T}\widehat{\mathbf{M}}_{\mathrm{FP}}\mathbf{T}^H + \mathbf{T}^*\widehat{\mathbf{M}}_{\mathrm{FP}}^T\mathbf{T}^T\right]\right)\\
&= \frac{1}{2}\boldsymbol{\mathcal{J}}^{\dagger}\left(\mathbf{T}^T \otimes \mathbf{T}^H\right)\left[\left(\mathbf{T}^* \otimes \mathbf{T}\right) + \left(\mathbf{T} \otimes \mathbf{T}^*\right)\mathbf{K}_m\right]\widehat{\boldsymbol{\eta}}_{\mathrm{FP}}\\
&= \frac{1}{2}\boldsymbol{\mathcal{J}}^{\dagger}\left[\mathbf{I}_{m^2} + \mathbf{L}_{m^2}\mathbf{K}_m\right]\widehat{\boldsymbol{\eta}}_{\mathrm{FP}} \quad (\text{cf. } \textbf{Lemma 2})\\
&= \boldsymbol{\mathcal{J}}^{\dagger}\widehat{\boldsymbol{\eta}}_{\mathrm{FP}} \qquad (7)
\end{aligned}$$

where $\mathbf{K}_m \in \mathbb{R}^{m^2\times m^2}$ is the commutation matrix satisfying the relation $\mathbf{K}_m\mathrm{vec}\left(\mathbf{A}^T\right) = \mathrm{vec}\left(\mathbf{A}\right)$ [19]. Furthermore, the latter matrix also verifies the relation $\mathbf{L}_{m^2}\mathbf{K}_m = \mathbf{K}_m\mathbf{L}_{m^2}$. The PFP-estimate, $\widehat{\boldsymbol{\mu}}_{\mathrm{PFP}}$, satisfies clearly the constraint $\mathrm{Tr}\left[\boldsymbol{\mathcal{M}}\left(\widehat{\boldsymbol{\mu}}_{\mathrm{PFP}}\right)\right] = m$ since

$$\mathrm{Tr}\left[\boldsymbol{\mathcal{M}}\left(\widehat{\boldsymbol{\mu}}_{\mathrm{PFP}}\right)\right] = \mathrm{vec}\left(\mathbf{I}_{m^2}\right)^T\boldsymbol{\mathcal{J}}\widehat{\boldsymbol{\mu}}_{\mathrm{PFP}} = \mathrm{Tr}\left[\widehat{\mathbf{M}}_{\mathrm{FP}}\right] = m$$

The equation (7) can be interpreted as the solution of the Euclidean projection of Tyler's estimate

$$\widehat{\boldsymbol{\mu}}_{\mathrm{PFP}} = \arg\min_{\alpha,\boldsymbol{\mu}} \|\widehat{\boldsymbol{\eta}}_{\mathrm{FP}} - \alpha\boldsymbol{\mathcal{J}}\boldsymbol{\mu}\|^2 \text{ s.t. } \mathrm{Tr}\left[\boldsymbol{\mathcal{M}}\left(\boldsymbol{\mu}\right)\right] = m \qquad (8)$$

4.3. Links between the two approaches

In this section, we show that the R-RCOMET procedure yields the same result whatever the number of iterations, K which coincides with the PFP estimate.

For $K = 1$, the R-RCOMET estimate is obtained by

$$\widehat{\boldsymbol{\mu}}_1 \propto \left(\boldsymbol{\mathcal{J}}^H\widehat{\mathbf{W}}_0^{-1}\boldsymbol{\mathcal{J}}\right)^{-1}\boldsymbol{\mathcal{J}}^H\widehat{\mathbf{W}}_0^{-1}\widehat{\boldsymbol{\eta}}_{\mathrm{FP}}$$

where $\widehat{\mathbf{W}}_0 = \boldsymbol{\mathcal{M}}\left(\widehat{\boldsymbol{\mu}}_0\right)^T \otimes \boldsymbol{\mathcal{M}}\left(\widehat{\boldsymbol{\mu}}_0\right)$ and $\widehat{\boldsymbol{\mu}}_0$ is given by (5). According to the **Proposition 1.** in the Appendix, $\widehat{\boldsymbol{\mu}}_1$ can be rewritten as

$$\begin{aligned}
\widehat{\boldsymbol{\mu}}_1 &\propto \boldsymbol{\mathcal{J}}^{\dagger}\widehat{\mathbf{W}}_0\boldsymbol{\mathcal{J}}^{\dagger^H}\boldsymbol{\mathcal{J}}^H\widehat{\mathbf{W}}_0^{-1}\widehat{\boldsymbol{\eta}}_{\mathrm{FP}}\\
&\propto \frac{1}{2}\boldsymbol{\mathcal{J}}^{\dagger}\widehat{\mathbf{W}}_0\left[\mathbf{I}_{m^2} + \mathbf{L}_{m^2}\mathbf{K}_m\right]\widehat{\mathbf{W}}_0^{-1}\widehat{\boldsymbol{\eta}}_{\mathrm{FP}} \quad (\text{cf. } \textbf{Lemma 2})\\
&\propto \frac{1}{2}\boldsymbol{\mathcal{J}}^{\dagger}\left[\mathbf{I}_{m^2} + \mathbf{L}_{m^2}\mathbf{K}_m\right]\widehat{\boldsymbol{\eta}}_{\mathrm{FP}} = \widehat{\boldsymbol{\mu}}_{\mathrm{PFP}}
\end{aligned}$$

Hence, to verify $\mathrm{Tr}\left[\boldsymbol{\mathcal{M}}\left(\widehat{\boldsymbol{\mu}}_1\right)\right] = m$, necessarily we have

$$\widehat{\boldsymbol{\mu}}_1 = \boldsymbol{\mathcal{J}}^{\dagger}\widehat{\boldsymbol{\eta}}_{\mathrm{FP}} = \widehat{\boldsymbol{\mu}}_{\mathrm{PFP}} \qquad (9)$$

By recurrence, we can easily show that $\widehat{\boldsymbol{\mu}}_1 = \widehat{\boldsymbol{\mu}}_K = \widehat{\boldsymbol{\mu}}_{\mathrm{PFP}}$ for any $K \in \mathbb{N}^*$. Consequently, the R-RCOMET procedure converges in only one step, i.e. the resulting outcome is identical for any $K \geq 1$.

Consequently, we have shown, for Hermitian persymmetric matrices, that RCOMET based procedure, which is based on the extended invariance principle, coincides with the classical Euclidean projection.

5. NUMERICAL RESULTS

In this section, we illustrate the results of the previous analysis. For $m = 8$, we generate 5000 sets of N independent m-dimensional centered t-distributed samples with $d = 5$ degrees of freedom. The texture for the t-distribution has the same distribution as $\tau \overset{d}{=} \dfrac{d}{x}$, where $x \sim \chi^2_d$.

We compare the performance of the previously studied algorithms: RCOMET and its recursive version R-RCOMET from [14], the PFP estimate derived in [11] and the Persymmetric Sample Covariance Matrix (PSCM) estimator. The latter is obtained with (6) by substituting the Tyler's estimate with the SCM. As explained in [14], the RCOMET based methods are efficient (i.e. the mean square error reaches the Cramér rao bound) w.r.t. the normalized data, $\mathbf{z}_n = \dfrac{\mathbf{y}_n}{\|\mathbf{y}_n\|} \sim \mathcal{U}\left(\mathbf{M}_e\right)$. Therefore due to the constraint on $\widehat{\boldsymbol{\mu}}$, we compare with the constrained CRB of the normalized data, $\mathbf{CRB}_{\mathcal{U}}$ [20, 21].

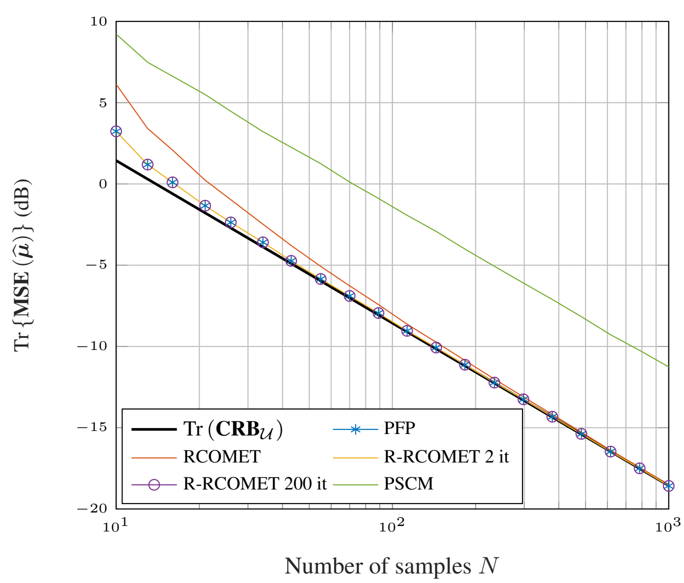

Fig. 1. Comparison on the MSE

From Fig. 1, we notice that the R-RCOMET estimates are identical for 2 and 200 iterations and also coincide with the PFP estimate. The PSCM estimator does not perform well since the SCM is not well-suited to non-Gaussian distributions.

6. CONCLUSION

In this paper, we have adressed the connections between different estimators of the shape matrix of CG distributed observations and having a Hermitian persymmetric structure. Although theses estimators have been derived originally from distinct methodologies, we have shown that some of them are surprisingly equivalent.

APPENDIX
USEFUL PROPERTIES AND RELATIONS

In the appendix, some properties are presented and demonstrated, which are required in the section 4.

Lemma 1. Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ be a Hermitian persymmetric matrix. Then, the matrices $\mathbf{A}^{-1}$, $\mathbf{W} = \mathbf{A}^T \otimes \mathbf{A}$ are also persymmetric.

Proof. Since $\mathbf{A}$ is persymmetric, we have $\mathbf{A} = \mathbf{L}_m\mathbf{A}^T\mathbf{L}_m$ and $\mathbf{A}^{-1} = \left(\mathbf{L}_m\mathbf{A}^T\mathbf{L}_m\right)^{-1} = \mathbf{L}_m^{-1}\mathbf{A}^{-T}\mathbf{L}_m^{-1} = \mathbf{L}_m\left(\mathbf{A}^{-1}\right)^T\mathbf{L}_m$, because $\mathbf{L}_m^{-1} = \mathbf{L}_m$. Hence, $\mathbf{A}^{-1}$ is a persymmetric matrix. Then, we verify easily that

$$\begin{aligned}\mathbf{L}_{m^2}\mathbf{W}^T\mathbf{L}_{m^2} &= (\mathbf{L}_m\otimes\mathbf{L}_m)\left(\mathbf{A}\otimes\mathbf{A}^T\right)(\mathbf{L}_m\otimes\mathbf{L}_m)\\ &= (\mathbf{L}_m\mathbf{A}\mathbf{L}_m)\otimes\left(\mathbf{L}_m\mathbf{A}^T\mathbf{L}_m\right) = \mathbf{W}\end{aligned}$$

■

Lemma 2. Let $\boldsymbol{\mathcal{J}} \in \mathbb{C}^{m^2\times p}$ be the matrix defined in (3). Its left inverse is the matrix $\boldsymbol{\mathcal{J}}^\dagger \in \mathbb{C}^{p\times m^2}$ such as $\boldsymbol{\mathcal{J}}^\dagger = \left(\boldsymbol{\mathcal{J}}^H\boldsymbol{\mathcal{J}}\right)^{-1}\boldsymbol{\mathcal{J}}^H$ and $\boldsymbol{\mathcal{J}}^\dagger\boldsymbol{\mathcal{J}} = \mathbf{I}_p$. Then, we have the equalities

$$\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{J}}^\dagger = \frac{1}{2}\left(\mathbf{I}_{m^2} + \mathbf{L}_{m^2}\mathbf{K}_m\right) \quad (10)$$

$$\text{and} \qquad \boldsymbol{\mathcal{J}} = \mathbf{L}_{m^2}\mathbf{K}_m\boldsymbol{\mathcal{J}} \quad (11)$$

Proof. Let us introduce $\mathbf{P}_{\boldsymbol{\mathcal{J}}} = \boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{J}}^\dagger = \boldsymbol{\mathcal{J}}\left(\boldsymbol{\mathcal{J}}^H\boldsymbol{\mathcal{J}}\right)^{-1}\boldsymbol{\mathcal{J}}^H$ and $\mathbf{Q} = \frac{1}{2}\left(\mathbf{I}_{m^2} + \mathbf{L}_{m^2}\mathbf{K}_m\right)$.
On the one hand, $\mathbf{P}_{\boldsymbol{\mathcal{J}}}$ is a projection matrix onto the image of $\boldsymbol{\mathcal{J}}$ and has a rank equal to $p = \dfrac{m(m+1)}{2}$. On the other hand, $\mathbf{Q}$ is also a projection matrix onto the image of $\boldsymbol{\mathcal{J}}$, i.e. $\mathbf{Q}\mathbf{Q} = \mathbf{Q}$, $\mathbf{Q} = \mathbf{Q}^H$ and $\mathbf{Q}\boldsymbol{\mathcal{J}} = \boldsymbol{\mathcal{J}}$. Furthermore, we have $\mathbf{Q}\mathbf{P}_{\boldsymbol{\mathcal{J}}} = \mathbf{P}_{\boldsymbol{\mathcal{J}}}$. To conclude that $\mathbf{Q} = \mathbf{P}_{\boldsymbol{\mathcal{J}}}$, we need to show that $\text{rank}\,(\mathbf{Q}) = p$.
Since $\mathbf{K}_m$ is full rank, we have

$$\text{rank}\,(\mathbf{Q}) = \text{rank}\,(\mathbf{Q}\mathbf{K}_m) = \text{rank}\,(\mathbf{K}_m + \mathbf{L}_{m^2})$$

Let us analyse the columns of the matrix $\mathbf{B} = \mathbf{K}_m + \mathbf{L}_{m^2} \in \mathbb{R}^{m^2\times m^2}$. To this end, let be $\mathbf{e}_i$ the i-th column unit vector of order m^2 and $\mathbf{H}_{k,\ell} \in \mathbb{R}^{m\times m}$ the matrix with a 1 in its (k,ℓ) element and zeros elsewhere. We have

$$\begin{aligned}&\mathbf{L}_{m^2}\mathbf{e}_i = \mathbf{e}_{m^2-i+1} \quad \forall\, i \in [\![1;m^2]\!]\\ &\mathbf{K}_m\text{vec}\,(\mathbf{H}_{k,\ell}) = \mathbf{K}_m\mathbf{e}_{k+(\ell-1)m} = \text{vec}\left(\mathbf{H}_{k,\ell}^T\right) = \mathbf{e}_{\ell+(k-1)m}\end{aligned}$$

Thus, for $(k,\ell) \in [\![1;m^2]\!]^2$, we have for $i = k+(\ell-1)m \in [\![1;m^2]\!]$

$$\mathbf{B}\mathbf{e}_i = \mathbf{e}_{\ell+(k-1)m} + \mathbf{e}_{m^2-i+1} = \mathbf{e}_{\ell+(k-1)m} + \mathbf{e}_{m(m-\ell)+m-k+1}$$

The columns of $\mathbf{B}$ have either only one non-zero coordinate, equal to 2 (case 1.), or only two non-zero coordinates, equals to 1 (case 2.).

case 1. for $(k,\ell) \in [\![1;m^2]\!]^2$, there is m possibilities, all different, to have $\mathbf{e}_{\ell+(k-1)m} = \mathbf{e}_{m(m-\ell)+m-k+1}$, so $\mathbf{B}$ has at least m independent columns.

case 2. for $(k,\ell) \in [\![1;m^2]\!]^2\setminus\{(k,\ell)|k+\ell = m+1\}$, there is m^2-m possibilities, but only $\dfrac{m(m-1)}{2}$ different. Indeed, the associated couple $(k_1,\ell_1) = (m-\ell+1, m-k+1)$ has the same result as (k,ℓ), by pairs.

Finally, we obtain $\text{rank}\,(\mathbf{Q}) = \text{rank}\,(\mathbf{B}) = m + \dfrac{m(m-1)}{2} = p$, hence the relation (10).
The equation (11) arises straightforwardly from (10) by right multiplying (10) with $\boldsymbol{\mathcal{J}}$. ■

Proposition 1. Let be $\mathbf{W} = \mathbf{A}^T\otimes\mathbf{A}$, where $\mathbf{A} \in \mathbb{C}^{m\times m}$ is persymmetric. Then the inverse of the matrix $\boldsymbol{\mathcal{J}}^H\mathbf{W}^{-1}\boldsymbol{\mathcal{J}}$ is $\boldsymbol{\mathcal{J}}^\dagger\mathbf{W}\boldsymbol{\mathcal{J}}^{\dagger^H}$.

Proof. Let us introduce $\mathbf{X} = \boldsymbol{\mathcal{J}}^H\mathbf{W}^{-1}\boldsymbol{\mathcal{J}}$ and $\mathbf{Y} = \boldsymbol{\mathcal{J}}^\dagger\mathbf{W}\boldsymbol{\mathcal{J}}^{\dagger^H}$. We must show that $\mathbf{XY} = \mathbf{YX} = \mathbf{I}_p$. First, we have

$$\begin{aligned}\mathbf{XY} &= \boldsymbol{\mathcal{J}}^H\mathbf{W}^{-1}\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{J}}^\dagger\mathbf{W}\boldsymbol{\mathcal{J}}^{\dagger^H}\\ &= \frac{1}{2}\boldsymbol{\mathcal{J}}^H\left[\mathbf{I}_{m^2} + \mathbf{W}^{-1}\mathbf{L}_{m^2}\mathbf{K}_m\mathbf{W}\right]\boldsymbol{\mathcal{J}}^{\dagger^H} \ \text{(cf. \textbf{Lemma 2})}\end{aligned}$$

According to **Lemma 1**, the matrix $\mathbf{W}^{-1}$ is persymmetric. Thus, we have

$$\mathbf{W}^{-1}\mathbf{L}_{m^2}\mathbf{K}_m\mathbf{W} = \mathbf{L}_{m^2}\mathbf{W}^{-T}\mathbf{K}_m\mathbf{W} = \mathbf{L}_{m^2}\mathbf{W}^{-T}\mathbf{W}^T\mathbf{K}_m = \mathbf{L}_{m^2}\mathbf{K}_m$$

Finally, we obtain

$$\mathbf{XY} = \frac{1}{2}\boldsymbol{\mathcal{J}}^H\left[\mathbf{I}_{m^2} + \mathbf{L}_{m^2}\mathbf{K}_m\right]\boldsymbol{\mathcal{J}}^{\dagger^H} = \boldsymbol{\mathcal{J}}^H\left(\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{J}}^\dagger\right)^H\boldsymbol{\mathcal{J}}^{\dagger^H} = \mathbf{I}_p$$

The equality $\mathbf{YX} = \mathbf{I}_p$ is demonstrated analogously. ■

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