Radar Detection Schemes for Joint Time and Spatial Correlated Clutter Using Random Matrix Theory

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Motivations

Adaptive radar detection and estimation schemes are often based on the independence of the secondary data used for building estimators and detectors. This independence allows to build Likelihood functions.

Example: estimating a covariance matrix $\mathbf{M}$

With a given set of K independent N-dimensional vectors $\{\mathbf{y}_i\}_{i\in[1,K]}$ distributed according to $\mathcal{CN}(\mathbf{0}_N, \mathbf{M})$, the corresponding Likelihood function Λ can be built as

$$\Lambda(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_K \,|\, \mathbf{M}) = \prod_{i=1}^{K} p(\mathbf{y}_i) = \prod_{i=1}^{K} \frac{1}{\pi^N |\mathbf{M}|} \exp\left(-\mathbf{y}_i^H \mathbf{M}^{-1} \mathbf{y}_i\right).$$

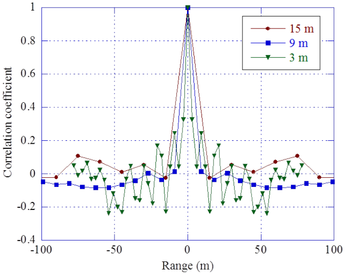
The Maximum Likelihood Estimate $\widehat{\mathbf{M}}$ of $\mathbf{M}$ is the zero of the partial derivative of $\Lambda(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_K \,|\, \mathbf{M})$ with respect to $\mathbf{M}$ leading to the well known SCM:

$$\widehat{\mathbf{M}} = \frac{1}{K} \sum_{i=1}^{K} \mathbf{y}_i \mathbf{y}_i^H.$$

Motivations

In many radar and imagery applications, data $\{\mathbf{y}_i\}_{i\in[1,K]}$ can be viewed as a joint spatial and temporal process:

- For high resolution radar, the sea clutter is clearly jointly spatially and temporally correlated,



Sea clutter spatial correlation, IPIX radar [M.S. Greco *et al*, 2006].

Motivations

- In mutichannel (polarimetric, interferometric or multi-temporal) SAR imaging, the multivariate vector characterizing each spatial pixel of the image is correlated over the channels but can also be strongly correlated with those of neighbourhood pixels,
- When a radar signal with bandwidth B is oversampled ($Fe = k\,B$, $k > 1$), the associated range bins can be spatially correlated and the measurements are not independent anymore.

In the radar community, one generally supposes that the vectors of information collected over a spatial support are identically and independently distributed.

This problem could be, for example, adressed using Multidimensional Space-time ARMA modeling (forthcoming PostDoc in SONDRA in next Sept. in collaboration with P. Bondon (L2S CentraleSupélec)).

The aim of this work is to relax this hypothesis through the use of recent Random Matrix Theory results.

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Problem formulation

Detection of a complex signal corrupted by an additive Gaussian noise $\mathbf{c} \sim \mathcal{CN}(\mathbf{0}_N, \mathbf{M})$ in a N-dimensional complex observation vector $\mathbf{y}$:

$$\begin{cases} H_0 : \mathbf{y} = \mathbf{c} & \mathbf{y}_i = \mathbf{c}_i \quad i = 1, \ldots, K \\ H_1 : \mathbf{y} = \alpha\, \mathbf{p} + \mathbf{c} & \mathbf{y}_i = \mathbf{c}_i \quad i = 1, \ldots, K \end{cases},$$

where $\mathbf{p}$ is a perfectly known complex steering vector, α is the unknown signal amplitude and where the $\mathbf{c}_i \sim \mathcal{CN}(\mathbf{0}_N, \mathbf{M})$ are K signal-free non independent measurements. The covariance matrix $\mathbf{M}$ characterizes the temporal or spectral correlation within the components of the noise vectors.

To model the spatial dependency between the secondary data, from the Gaussian assumption on $\mathbf{c}_i$, we may write the $N \times K$-matrix $\mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_K]$ under the following form:

$$\mathbf{C} = \mathbf{M}^{1/2}\, \mathbf{X} \mathbf{T}^{1/2},$$

where $\mathbf{M} \in \mathbb{C}^{N \times N}$ and $\mathbf{T} \in \mathbb{C}^{K \times K}$ are both nonnegative definite, $\mathbf{X}$ is standard Gaussian $\mathcal{CN}(\mathbf{0}_N, \mathbf{I}_N)$, and where $\mathbf{T}$ satisfies the normalization $\dfrac{1}{K}\mathrm{tr}\ \mathbf{T} = 1$.

Problem formulation

The matrix $\mathbf{T}$ is considered Toeplitz, i.e., for all i, j, $\mathbf{T}_{i,j} = t_{|i-j|}$ for $t_0 = 1$ and $t_k \in \mathbb{C}$, and positive definite. Besides, $\sum_{k=0}^{K-1} |t_k| < \infty$.

Example: $N = 2$, $K = 3$

$$\mathbf{C} = \underbrace{\underbrace{\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}^{1/2}}_{\text{Temporal correlation}} \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \end{pmatrix}}_{\text{Temporal or Spectral Measurements}} \underbrace{\begin{pmatrix} t_0 & t_1 & t_2 \\ t_1 & t_0 & t_1 \\ t_2 & t_1 & t_0 \end{pmatrix}^{1/2}}_{\text{Spatial correlation}}.$$

Outline

Key ideas

Let $\{\mathbf{c}_i\}_{i\in[1,K]}$ be distributed according to $\mathcal{CN}\left(\mathbf{0}_N, \mathbf{M}\right)$. The Maximum Likelihood Estimate of $\mathbf{M}$ is given by $\widehat{\mathbf{M}} = \dfrac{1}{K}\displaystyle\sum_{i=1}^{K} \mathbf{c}_i\, \mathbf{c}_i^H = \dfrac{1}{K}\mathbf{C}\,\mathbf{C}^H$.

Asymptotic Regime

If $K \to \infty$, then the strong law of large numbers says (or equivalently, in spectral norm):

$$\widehat{\mathbf{M}} - \mathbf{M} \xrightarrow{a.s.} \mathbf{0}_N, \qquad \left|\left|\widehat{\mathbf{M}} - \mathbf{M}\right|\right| \xrightarrow{a.s.} 0\,.$$

Random Matrix Regime

- No longer valid if N, $K \to \infty$ with $N/K \to c \in [0,\infty[$: $\left|\left|\widehat{\mathbf{M}} - \mathbf{M}\right|\right| \nrightarrow 0$,
- For practical large N, K with $N \simeq K$, it can lead to dramatically wrong conclusions (even $N = K/100$).

RMT in SONDRA for Radar, SAR and Hyperspectral Applications

The RMT is not a magic tool or *scientific elucubration* but allows 1) to understand the statistical behavior of expressions involving estimate of large covariance matrices (ex: quadratic forms, ratios of the quadratic forms, SNIR Loss, performances of detection tests as ANMF, LR-ANMF, etc.) and 2) to correct it. At a finite distance (practical N, K values), the corrected results are often valid.

- Sources localisation applications [F. Pascal, R. Couillet, ...]: the based-RMT Music algorithm (G-Music) is known to have higher performance than those of conventional algorithms when using all the eigenvalues of the covariance matrix.
- MIMO-STAP: the goal of A. Combernoux PHD thesis was to analyse/improve the detection and filtering performances of low-rank detectors.
- Hyperspectral Anomaly Detection - Unmixing: the goal of E. Terreaux PhD thesis is to better analyse the rank of the anomalies space in Hyperspectral Imaging for heterogeneous and non-Gaussian environment.

Some RMT results

Proposition: Consistent Estimation for $\mathbf{T}$ [R. Couillet *et al*, 2015]

As $N, K \to \infty$ *such that* $N/K \to c \in [0, \infty[$, *and for every* $\beta < 1$,

$$N^{\beta} \left\| \mathcal{T}\left[\frac{1}{N}\mathbf{C}^H\mathbf{C}\right] - \left(\frac{1}{N}\operatorname{tr}\mathbf{M}\right)\mathbf{T} \right\|_F \xrightarrow{a.s.} 0\,,$$

where $\mathcal{T}[\cdot]$ *is the Toeplitzification operator:* $(\mathcal{T}[\mathbf{X}])_{ij} = \dfrac{1}{K}\displaystyle\sum_{k=1}^{K} \mathbf{X}_{k,k+|i-j|}$.

Up to a constant, a consistent estimator $\hat{\mathbf{T}}$ of the spatial covariance $\mathbf{T}$ characterizing data $\{\mathbf{c}_i\}_{i\in[1,K]}$ is therefore defined as $\hat{\mathbf{T}} \propto \mathcal{T}\left[\frac{1}{N}\mathbf{C}^H\mathbf{C}\right]$ and the associated time whitened sample covariance matrix estimate $\widehat{\mathbf{M}}$ of $\mathbf{M}$ is defined as $\widehat{\mathbf{M}} \propto \frac{1}{K}\mathbf{C}\,\hat{\mathbf{T}}^{-1}\mathbf{C}^H$.

This technique has been extended in the framework of robust M-estimators.

Outline

The two-step GLRT ANMF

ANMF test (ACE, GLRT-LQ) [Conte, 1995]

$$\Lambda_{ANMF}(\mathbf{y}, \widehat{\mathbf{M}}) = \frac{|\mathbf{p}^H \widehat{\mathbf{M}}^{-1} \mathbf{y}|^2}{(\mathbf{p}^H \widehat{\mathbf{M}}^{-1} \mathbf{p})(\mathbf{y}^H \widehat{\mathbf{M}}^{-1} \mathbf{y})} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda,$$

where $\widehat{\mathbf{M}}$ stands for any covariance matrix estimators.

The ANMF is scale-invariant (homogeneous of degree 0), i.e.

$$\forall \alpha, \beta \in \mathbb{R}, \; \Lambda_{ANMF}(\alpha\, \mathbf{y}, \beta\, \widehat{\mathbf{M}}) = \Lambda_{ANMF}(\mathbf{y}, \widehat{\mathbf{M}}).$$

- In Gaussian distributed clutter, we can use practically Sample Covariance Matrix approach: $\widehat{\mathbf{M}} = \dfrac{1}{K} \displaystyle\sum_{i=1}^{K} \mathbf{c}_i\, \mathbf{c}_i^H = \dfrac{1}{K} \mathbf{C}\, \mathbf{C}^H$,
- In more complex SIRV or CES distributed clutter, we can use robust $\mathbf{M}$-estimators approach (e.g. Tyler): $\widehat{\mathbf{M}} = \dfrac{N}{K} \displaystyle\sum_{i=1}^{K} \dfrac{\mathbf{c}_i\, \mathbf{c}_i^H}{\mathbf{c}_i^H\, \widehat{\mathbf{M}}^{-1}\, \mathbf{c}_i^H}$.

Gaussian and non-Gaussian scenarios

Simulated Data: joint spatial and time correlated Gaussian or K-distributed ($\nu = 0.5$) data characterized by $N = 10$ pulses, $K = 20$ secondary data where:

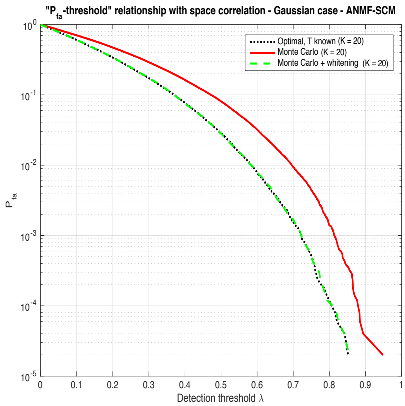
$$\mathbf{M} = \left(\rho_{\mathbf{M}}^{|i-j|}\right)_{i,j\in[1,N]},\ \mathbf{T} = \left(\rho_{\mathbf{T}}^{|i-j|}\right)_{i,j\in[1,K]} \text{ with } \rho_{\mathbf{M}} = 0.5,\ \rho_{\mathbf{T}} = 0.9.$$

To evaluate the detection performance of the Λ_{ANMF} test statistic, we have compared three approaches:

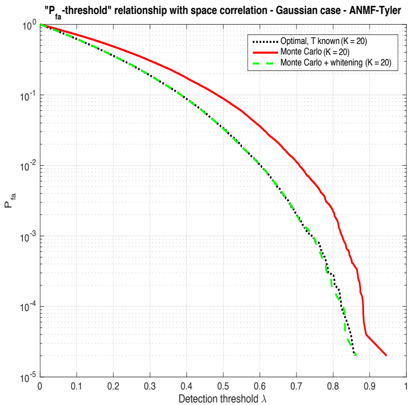
- $\mathbf{M}$ is unknown but $\mathbf{T}$ is assumed to be known: the covariance estimate $\widehat{\mathbf{M}}$ is either given by $\frac{1}{K}\mathbf{C}\,\mathbf{T}^{-1}\,\mathbf{C}^H$ (SCM) or the Tyler's estimate of the true spatial-whitened data $\mathbf{C}\,\mathbf{T}^{-1/2}$,
- $\mathbf{T}$ is assumed to be unknown and is estimated through $\hat{\mathbf{T}} \propto \mathcal{T}\left[\frac{1}{N}\,\mathbf{C}^H\,\mathbf{C}\right]$: the covariance estimate $\widehat{\mathbf{M}}$ is either given by $\frac{1}{K}\,\mathbf{C}\,\hat{\mathbf{T}}^{-1}\,\mathbf{C}^H$ (SCM) or the Tyler's estimate of the spatial-whitened data $\mathbf{C}\,\hat{\mathbf{T}}^{-1/2}$,
- the classical approach that does not take into account the space correlation: the covariance estimate $\widehat{\mathbf{M}}$ is either given by $\frac{1}{K}\mathbf{C}\,\mathbf{C}^H$ (SCM) or Tyler's estimate of the data $\mathbf{C}$.

False Alarm Regulation - Gaussian Case

ANMF-SCM



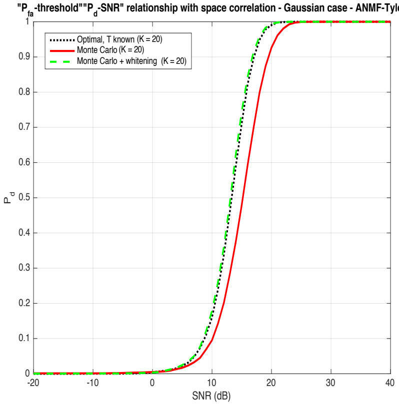
ANMF-Tyler



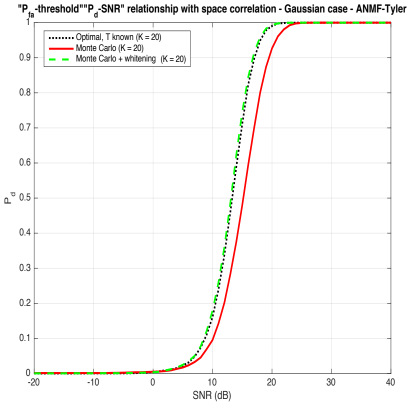
Same False Alarm Regulation performance for ANMF-SCM and ANMF-Tyler (Gaussian case)

Associated Detection Performance - Gaussian Case

ANMF-SCM



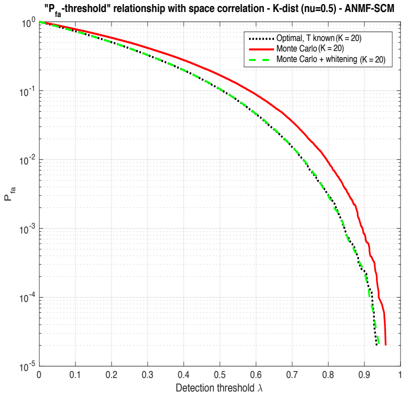
ANMF-Tyler



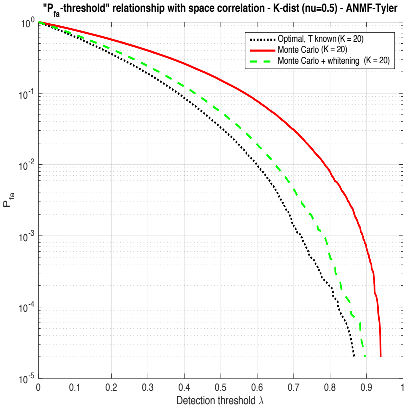
- Same Probability of Detection performance.
- Around 3dB gain improvement with RMT whitening procedure

False Alarm Regulation - K-distributed Case

ANMF-SCM



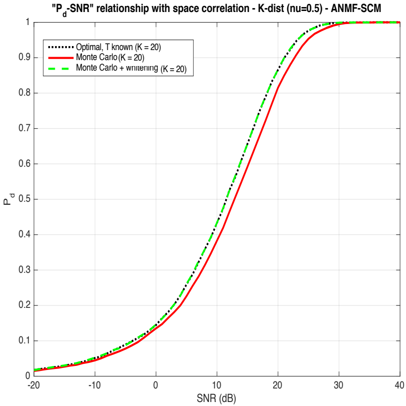
ANMF-Tyler



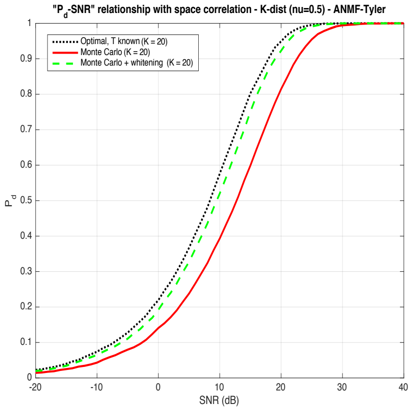
- Better False Alarm regulation performance for ANMF-FP (Non-Gaussian case).
- Better False Alarm regulation with RMT whitening procedure

Associated Detection Performance - K-distributed Case

ANMF-SCM



ANMF-Tyler



- Better performances in terms of Probability of Detection performance for ANMF-Tyler.
  - Around 3dB gain improvement with RMT whitening procedure

Outline

Conclusions

This work has focused on the joint estimation of joint spatial and temporal covariance matrices arising for adaptive radar detection schemes:

- This estimation was efficiently performed using latest results coming from RMT with a Toeplitz covariance structure assumption for the spatial covariance matrix,
- First results show that the ANMF built with these new estimates has significant higher performances, in terms of regulation of false alarm and probability of detection versus SNR, than those of the ANMF built with classical estimates supposing erroneously i.i.d. spatially secondary data,
- This RMT technique has been extended for heterogeneous and Non-Gaussian environment. $\mathbf{M}$-estimators taking into account spatial correlation have similar performance as those of SCM in Gaussian environment but outperform them in non-Gaussian environment,
- This quite *simple* technique can be easilly applied on experimental data (radar, STAP, MIMO-STAP, SAR, HS).