Recent Advances in Adaptive Radar Detection

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Part B

Adaptive Detection and Covariance Matrix Estimation

Part B: Contents

Key references of Part B

- M. Mahot, F. Pascal, J-P. Ovarlez and P. Forster, "Asymptotic properties of robust complex covariance matrix estimates," *Signal Processing, IEEE Transactions on,*, vol. 61, pp. 3348-3356, July 2013.
- E. Ollila, D. E. Tyler, V. Koivunen and H.V. Poor, "Complex elliptically symmetric distributions: survey, new results and applications," *Signal Processing, IEEE Transactions on,* vol. 60, no. 11, pp. 5597 - 5625, Nov. 2012.
- F. Gini, A. Farina, and M. V. Greco, "Selected list of references on radar signal processing," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 37, pp. 329 - 359, January 2001.

Outline

Motivations

- Application reality: only observations $\Rightarrow$ Unknown parameters
- Several SP applications require the covariance matrix estimation, e.g. sources localization, STAP, Polarimetric SAR classification, radar detection, MIMO...
- The ultimate purpose is to characterize the system performance, not only the estimation performance $\Rightarrow$ ROC curves, probability of detection vs SNR, false alarm regulation, MSE characterization...

Outline

Reminders: Problem Statement

- In a m-vector $\mathbf{z}$, detecting a complex known signal $\mathbf{s} = A\,\mathbf{p}$ embedded in an additive noise $\mathbf{y}$ (with covariance matrix Σ) , can be written as the following statistical test:

$$\begin{cases} \text{Hypothesis } H_0\text{:} & \mathbf{z} = \mathbf{y} & \mathbf{z}_i = \mathbf{y}_i & i = 1, \ldots, n \\ \text{Hypothesis } H_1\text{:} & \mathbf{z} = \mathbf{s} + \mathbf{y} & \mathbf{z}_i = \mathbf{y}_i & i = 1, \ldots, n \end{cases}$$

where the $\mathbf{z}_i$'s are n "signal-free" independent observations (secondary data) used to estimate the noise parameters .

$\Rightarrow$ Neyman-Pearson criterion

Reminder: Detection generalities

- Detection test: comparison between the Likelihood Ratio $\Lambda(\mathbf{z})$ and a detection threshold λ:

$$\Lambda(\mathbf{z}) = \frac{p_{\mathbf{z}}(\mathbf{z}/H_1)}{p_{\mathbf{z}}(\mathbf{z}/H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda\,,$$

λ is obtained for a given *PFA* (set by the user):

- Probability of False Alarm (type-I error):

$$PFA = \mathbb{P}(\Lambda(\mathbf{z}) > \lambda/H_0)$$

- Probability of Detection (to evaluate the performance):

$$PD = \mathbb{P}(\Lambda(\mathbf{z}) > \lambda/H_1)$$

for different Signal-to-Noise Ration (SNR).

Reminder: Gaussian/non-Gaussian assumptions

- Gaussian case (OGD): if $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}, \Sigma)$ then

$$\Lambda(\mathbf{y}) = \frac{|\mathbf{p}^H \Sigma^{-1} \mathbf{z}|^2}{\mathbf{p}^H \Sigma^{-1} \mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_g$$

with $\lambda_g = \sqrt{-\ln(PFA)}$.

- Heterogeneous case (NMF):

$$\Lambda(\mathbf{y}) = \frac{|\mathbf{p}^H \Sigma^{-1} \mathbf{z}|^2}{(\mathbf{p}^H \Sigma^{-1} \mathbf{p})(\mathbf{z}^H \Sigma^{-1} \mathbf{z})} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{NMF}$$

The False Alarm regulation can be theoretically done thanks to

$$\lambda_{NMF} = 1 - PFA^{\frac{1}{m-1}}.$$

This comes from a Beta distribution of the test.

Going to adaptive detection

Generally, some parameters (say Σ!) are unknown.

$\Rightarrow$ Covariance Matrix Estimation

Requirements:

- Background modeling: Gaussian, SIRV, CES, K-distribution...
- Estimation procedure: ML-based approaches, M-estimation, Z-estimation, LS-based methods...
- Adaptive detectors and adaptive performance

Outline

Standard approches: Gaussian noise/clutter

The Sample Covariance Matrix (SCM)

$$\widehat{\mathbf{S}}_n = \frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i \mathbf{z}_i^H$$

where $\mathbf{z}_i$ are complex independent circular zero-mean Gaussian with covariance matrix Σ, i.e. $p_{\mathbf{z}_i}(\mathbf{z}_i) = \frac{1}{(\pi)^m |\Sigma|} \exp\left(-\mathbf{z}_i^H \, \Sigma^{-1} \, \mathbf{z}_i\right)$.

The Shrinkage or Diagonal Loading SCM

$$\widehat{\mathbf{S}}_{Sh.} = (1-\beta)\,\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i \mathbf{z}_i^H + \beta\,\mathbf{I} \qquad \text{or} \qquad \widehat{\mathbf{S}}_{DL} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i \mathbf{z}_i^H + \beta\,\mathbf{I}$$

Standard approches: Gaussian noise/clutter

Properties of the SCM

- Simple CM estimator
- Very tractable
- Well-known statistical properties: constant, unbiased and efficient

Then, $\sqrt{n}\,\mathrm{vec}(\widehat{\mathbf{S}}_n - \Sigma) \xrightarrow{d} \mathbb{C}\mathcal{N}\left(\mathbf{0}, \mathbf{C}, \mathbf{P}\right)$

where $\begin{array}{l} \mathbf{C} = \left(\Sigma^* \otimes \Sigma\right) \\ \mathbf{P} = \left(\Sigma^* \otimes \Sigma\right) \mathbf{K}_{m,m} \end{array}$

Outline

Motivations

Why non-Gaussian techniques? Examples in Radar processing

Classical radar applications consider the background to be Gaussian.

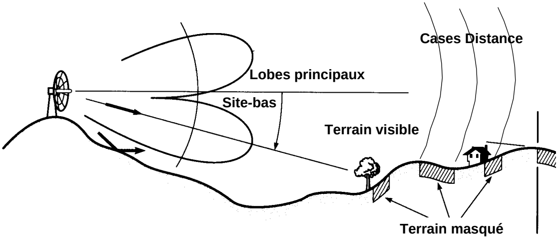
$\longrightarrow$ The Sample Covariance Matrix

- a simple estimator
- well-known statistical properties

Robustness: what happens in non-Gaussian models?

- High resolution techniques and/or low grazing angle radars
- Outliers and other parasites are not been taken into account with the Gaussian model.
- The SCM may give poor results.

- Grazing angle Radar



  - $\Rightarrow$ Impulsive Clutter
  - $\Rightarrow$ Spatial heterogeneity (e.g. in SAR or HS images)

- High Resolution Radar

  - $\Rightarrow$ Small number of scatters in the Cell Under Test (CUT)
  - $\Rightarrow$ Central Limit Theorem (CLT) is not valid anymore

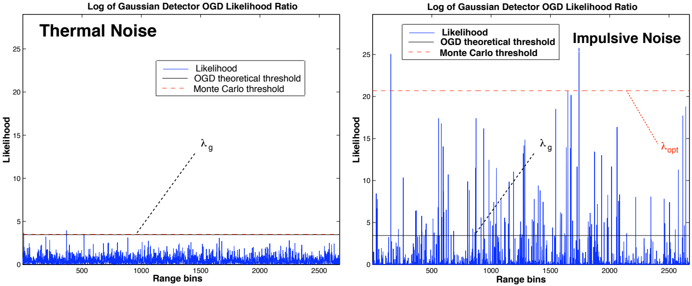


Figure: Failure of the OGD - Adjustment of the detection threshold - K-distributed clutter with same power as the Gaussian noise

$\Rightarrow$ Bad performance of the OGD in case of mismodeling

$\Rightarrow$ Need/Use of CES distributions

$\Rightarrow$ Need/Use of robust estimates

Outline

Modeling the background

Let $\mathbf{z}$ be a complex circular random vector of length m. $\mathbf{z}$ has a complex elliptically symmetric (CES) distribution ($CE(\boldsymbol{\mu}, \Sigma, g_{\mathbf{z}})$) if its PDF is

$$g_{\mathbf{z}}(\mathbf{z}) = |\Sigma|^{-1} h_z((\mathbf{z}-\boldsymbol{\mu})^H \Sigma^{-1} (\mathbf{z}-\boldsymbol{\mu})), \quad (1)$$

where $h_z : [0, \infty) \to [0, \infty)$ is the density generator and is such as (1) defines a pdf.

- $\boldsymbol{\mu}$ is the statistical mean (generally known or $= \mathbf{0}$)
- Σ the scatter matrix

In general (finite second-order moment), the CM $= \alpha \Sigma$ where α is known.

Attractive clutter modeling

Some important properties

- Large class of distributions: Gaussian, SIRV, MGGD, K-dist., Student-t....
- Closed under affine transformations
- All sub-vectors of $\mathbf{z}$ have a CES dist.
- Stochastic representation theorem

$\mathbf{z} \sim \mathsf{CE}_m(\boldsymbol{\mu}, \Sigma)$ iff it admits the stochastic representation:
$\boxed{\mathbf{z} =_d \boldsymbol{\mu} + \mathcal{R}\mathbf{A}\mathbf{u}^{(k)}}$ where $\mathcal{R} \geq 0$, independent of $\mathbf{u}^{(k)}$ and $\Sigma = \mathbf{A}\mathbf{A}^H$ is a factorisation of Σ, where $\mathbf{A} \in \mathbb{C}^{m \times k}$ with $k = \mathrm{rank}(\Sigma)$

Outline

Estimating the covariance matrix

M-estimators

PDF is specified $\Rightarrow$ MLE can be derived
PDF is not specified $\Rightarrow$ M-estimators are used instead

Let $(\mathbf{z}_1, ..., \mathbf{z}_n)$ be a n-sample $\sim CE_m(\mathbf{0}, \Sigma, g_\mathbf{z})$ (Secondary data).

M-estimator of Σ

$$\widehat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} u\left(\mathbf{z}_i^H \widehat{\Sigma}^{-1} \mathbf{z}_i^H\right) \mathbf{z}_i \mathbf{z}_i^H, \tag{2}$$

Maronna (1976), Kent and Tyler (1991)

- Existence
- Uniqueness
- Convergence of the recursive algorithm...

Examples of *M*-estimators

SCM:

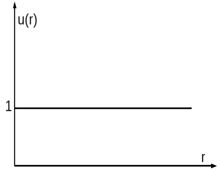
$u(r) = 1$

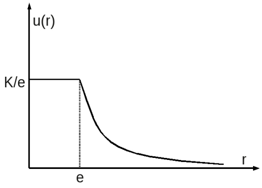
Huber's *M*-estimator:

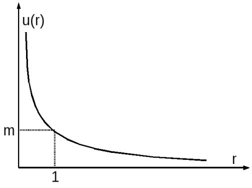
$$u(r) = \begin{cases} K/e \text{ if } r <= e \\ K/r \text{ if } r > e \end{cases}$$

FPE (Tyler):

$u(r) = \frac{m}{r}$







FPE and comments

Remarks:

- Huber = mix between SCM and FPE
- FPE and SCM are "not" (theoretically) *M*-estimators
- FPE is the most robust while SCM is the most efficient (in Gaussian case).

FP Estimate (Tyler, 1987; Pascal, 2008)

$$\widehat{\Sigma}_{FPE} = \frac{m}{n} \sum_{i=1}^{n} \frac{\mathbf{z}_i \mathbf{z}_i^H}{\mathbf{z}_i^H \widehat{\Sigma}_{FPE}^{-1} \mathbf{z}_i}$$

Properties of the *M*-estimators

Let us set

$$\mathbf{V} = E\left[u(\mathbf{z}'\mathbf{V}^{-1}\mathbf{z})\,\mathbf{z}\mathbf{z}'\right], \tag{3}$$

where $\mathbf{z} \sim CE(\mathbf{0}, \Sigma, g_{\mathbf{z}})$.

- (3) admits a unique solution $\mathbf{V}$ and $\mathbf{V} = \sigma\Sigma = \sigma/\alpha\,\mathbf{M}$ where σ is given by Tyler(1982),
- $\widehat{\Sigma}$ is a consistent estimate of $\mathbf{V}$.

Asymptotic distribution of complex M-estimators

Using the results of Tyler, we derived the following results (Mahot, 2013):

Theorem 1 (Asymptotic distribution of $\hat{\Sigma}$)

$$\sqrt{n}\,\text{vec}(\hat{\Sigma} - \Sigma) \xrightarrow{d} \mathbb{C}\mathcal{N}_{m^2}\left(\mathbf{0}, \mathbf{C}, \mathbf{P}\right), \tag{4}$$

where $\mathbb{C}\mathcal{N}$ is the complex Gaussian distribution, $\mathbf{C}$ the CM and $\mathbf{P}$ the pseudo CM:

$$\begin{aligned}\mathbf{C} &= \sigma_1(\Sigma^* \otimes \Sigma) + \sigma_2 \text{vec}(\Sigma)\text{vec}(\Sigma)^H,\\ \mathbf{P} &= \sigma_1(\Sigma^* \otimes \Sigma)\,\mathbf{K} + \sigma_2 \text{vec}(\Sigma)\text{vec}(\Sigma)^T,\end{aligned}$$

where $\mathbf{K}$ is the commutation matrix and where the constant σ_1 and σ_1 are completely defined.

Outline

An important property of complex M-estimators

- Let $\widehat{\Sigma}$ an estimate of Hermitian positive-definite matrix Σ that satisfies

$$\sqrt{n}\left(\mathrm{vec}(\widehat{\Sigma}-\Sigma)\right) \xrightarrow{d} \mathbb{C}\mathcal{N}\left(\mathbf{0},\mathbf{C},\mathbf{P}\right), \tag{5}$$

with

$$\left\{\begin{array}{l} \mathbf{C}=\nu_1\Sigma^*\otimes\Sigma+\nu_2\mathrm{vec}(\Sigma)\mathrm{vec}(\Sigma)^H, \\ \mathbf{P}=\nu_1(\Sigma^*\otimes\Sigma)\,\mathbf{K}_{m,m}+\nu_2\mathrm{vec}(\Sigma)\mathrm{vec}(\Sigma)^T, \end{array}\right.$$

where ν_1 and ν_2 are any real numbers.

e.g.

	SCM	M-estimators	FP
ν_1	1	σ_1	$(m+1)/m$
ν_2	0	σ_2	$-(m+1)/m^2$
...	More accurate		More robust

- Let $H(\mathbf{V})$ be a r-multivariate function on the set of Hermitian positive-definite matrices, with continuous first partial derivatives and such as $H(\mathbf{V}) = H(\alpha\mathbf{V})$ for all $\alpha > 0$, e.g. the ANMF statistic, the MUSIC statistic.

Theorem 2 (Asymptotic distribution of $H(\widehat{\Sigma})$)

$$\sqrt{n}\left(H(\widehat{\Sigma}) - H(\Sigma)\right) \xrightarrow{d} \mathbb{C}\mathcal{N}\left(\mathbf{0}_{r,1}, \mathbf{C}_H, \mathbf{P}_H\right) \tag{6}$$

where $\mathbf{C}_H$ and $\mathbf{P}_H$ are defined as

$$\mathbf{C}_H = \nu_1 H'(\Sigma)(\Sigma^T \otimes \Sigma) H'(\Sigma)^H,$$
$$\mathbf{P}_H = \nu_1 H'(\Sigma)(\Sigma^T \otimes \Sigma)\, \mathbf{K}_{m,m} H'(\Sigma)^T,$$

where $H'(\Sigma) = \left(\dfrac{\partial H(\Sigma)}{\partial \text{vec}(\Sigma)}\right)$.

Some comments:

Perfect (but asymptotic) characterization of several objects properties, such as detectors, classifiers, estimators...

$H(SCM)$ and $H(M$-estimators$)$ share the same asymptotic distribution (differs from σ_1)

$\Downarrow$

- Link to the classical Gaussian case
- Quantification of the loss involved by robust estimator

Adaptive Gaussian detection

$$\text{Gaussian model} \Rightarrow \widehat{\mathbf{S}}_n = \frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i \mathbf{z}_i^H$$

- AMF test [1]

$$\Lambda_{AMF}(\mathbf{y}) = \frac{\left|\mathbf{p}^H \, \widehat{\mathbf{S}}_n^{-1} \, \mathbf{y}\right|^2}{\left(\mathbf{p}^H \, \widehat{\mathbf{S}}_n^{-1} \, \mathbf{p}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{AMF} \,. \tag{7}$$

[1] F. C. Robey, D. R. Fuhrmann, E. J. Kelly, and R. Nitzberg, "A CFAR adaptive matched filter detector", *Aerospace and Electronic Systems, IEEE Transactions on*, vol. 28, no. 1, pp. 208-216, 1992.

- Kelly test [2]

$$\Lambda_{Kelly}(\mathbf{y}) = \frac{\left|\mathbf{p}^H \, \widehat{\mathbf{S}}_n^{-1} \, \mathbf{y}\right|^2}{\left(\mathbf{p}^H \, \widehat{\mathbf{S}}_n^{-1} \, \mathbf{p}\right) \, \left(N + \mathbf{y}^H \, \widehat{\mathbf{S}}_n^{-1} \, \mathbf{y}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{Kelly} \,. \tag{8}$$

[2] E. J. Kelly, "An adaptive detection algorithm", *Aerospace and Electronic Systems, IEEE Transactions on*, pp. 115-127, November 1986.

Outline

CES distribution $\Rightarrow$ ANMF

ANMF test (ACE, GLRT-LQ) [3,4]

$$\Lambda_{ANMF}(\mathbf{y}, \widehat{\Sigma}) = \frac{|\mathbf{p}^H \widehat{\Sigma}^{-1} \mathbf{y}|^2}{(\mathbf{p}^H \widehat{\Sigma}^{-1} \mathbf{p})(\mathbf{y}^H \widehat{\Sigma}^{-1} \mathbf{y})} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{ANMF} \tag{9}$$

where $\widehat{\Sigma}$ stands for any estimators presented before: SCM, M-estimators, Tyler's estimator...

One has, conditionally to $\mathbf{y}$, $\Lambda(\widehat{\Sigma}) = \Lambda(\alpha \widehat{\Sigma})$ for any $\alpha > 0$.

[3] E. Conte, M. Lops, and G. Ricci, "Asymptotically Optimum Radar Detection in Compound-Gaussian Clutter", *Aerospace and Electronic Systems, IEEE Transactions on,*, vol. 31, pp. 617-625, April 1995.
[4] S. Kraut and L. L. Scharf, "The CFAR adaptive subspace detector is a scale-invariant GLRT", *Signal Processing, IEEE Transactions on*, vol. 47, no. 9, pp. 2538-2541, 1999.

Properties

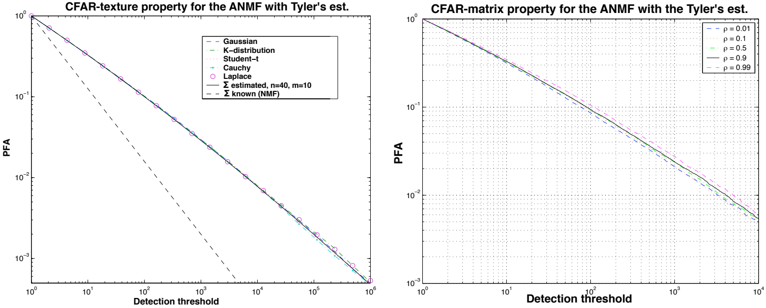
- The ANMF is scale-invariant, i.e.
  $\forall \alpha, \beta \in \mathbb{R}, \; \Lambda_{ANMF}(\alpha\, \mathbf{y}, \beta\, \widehat{\Sigma}) = \Lambda_{ANMF}(\mathbf{y}, \widehat{\Sigma})$
- Its asymptotic distribution (conditionally to $\mathbf{y}$!) is known (tks to theorem 2)

Considering $\Lambda_{ANMF}(\mathbf{y}, \widehat{\Sigma})$ conditionally to $\mathbf{y}$, i.e. $\Lambda_{ANMF}(\widehat{\Sigma})$, allows to directly apply theorem 2. Else see next slide!

- It is CFAR w.r.t the covariance/scatter matrix, i.e. its distribution does not depend on the covariance/scatter matrix
- It is CFAR w.r.t the texture (if considering Compound-Gaussian model)

Illustration of the CFAR properties

False Alarm regulation



(a) CFAR-texture (b) CFAR-matrix

Figure: Illustration of the CFAR properties of the ANMF built with the Tyler's estimator, for a Toeplitz CM whose (i, j)-entries are $\rho^{|i-j|}$

Probability of false alarm

PFA-threshold relation of $\Lambda_{ANMF}(\widehat{\mathrm{S}}_n)$ (Gaussian case, finite n)

$$P_{fa} = (1-\lambda)^{a-1}\,{}_2F_1(a, a-1; b-1; \lambda), \tag{10}$$

where $a = n - m + 2$, $b = n + 2$ and ${}_2F_1$ is the Hypergeometric function defined as

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{x^k}{k!} \tag{11}$$

[5] F. Pascal, J.-P. Ovarlez, P. Forster, and P. Larzabal, "Constant false alarm rate detection in spherically invariant random processes," in *Proc. of the European Signal Processing Conf., EUSIPCO-04*, (Vienna), pp. 2143-2146, Sept. 2004.

Comments

Three possible approaches to characterize the performance:

- Use the (very) poor approximation of the FA regulation of the NMF
- Use the asymptotics of theorem 2 (but it is conditionally to the dist. of $\mathbf{y}$!) $\Rightarrow$ a slight loss of performance
- Combine the asymptotics of theorem 9 of Part B and the finite-distance result on PFA-threshold...

From theorem 1 , one has

PFA-threshold relation of $\Lambda_{ANMF}(M\text{-est.})$ for CES distributions

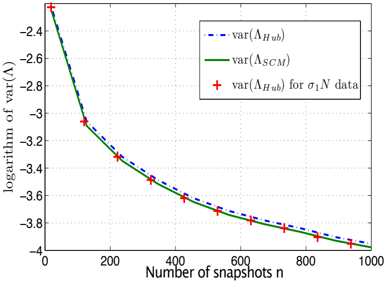
For n large enough and for any elliptically distributed noise, the PFA is still given by (10) if we replace n by n/σ_1.

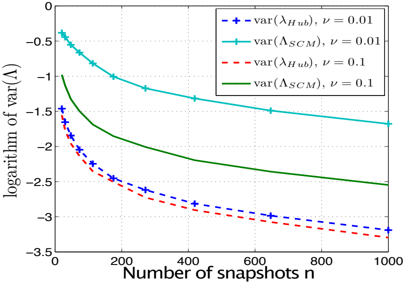
The third one seems to provide more accurate results...

Outline

Simulations

- Complex Huber's *M*-estimator.
- Figure 1: Gaussian context, here $\sigma_1 = 1.066$.
- Figure 2: K-distributed clutter (shape parameter: 0.1 and 0.01).



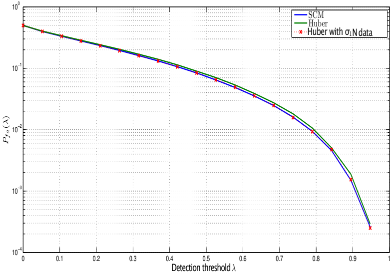


Validation of theorem (even for small n)

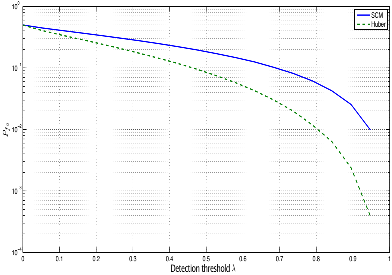
Interest of the *M*-estimators

Simulations: Probabilities of False Alarm

- Complex Huber's *M*-estimator.
- Figure 1: Gaussian context, here $\sigma_1 = 1.066$.
- Figure 2: K-distributed clutter (shape parameter: 0.1).



Validation of theorem (even for small *n*)

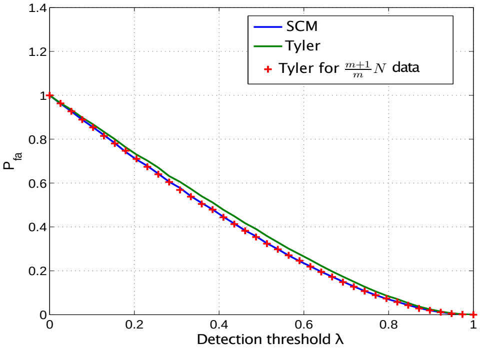


Interest of the *M*-estimators for False Alarm regulation

Tyler's estimator: Gaussian context, $n = 10$, $m = 3$

PFA-threshold relation of Λ_{ANMF}(Tyler's est.) for CES distributions

For n large and any elliptically distributed noise, the PFA is still given by (10) if we replace n by $n/\frac{m+1}{m}$.



Comments

Conclusions on the detection part:

Accurate approximation of the (theoretical) FA regulation

Cost: having a little bit more data: $\sigma_1\, n$ instead of n.

This σ_1 can be interpreted as the loss brought by robust estimators compared to optimal Gaussian estimator BUT performance stability of the robust estimators in various distributions contexts

Outline

Motivations

Some advantages

- Robustness to outliers
- May allow to include *a priori* informations
- Case of small number of observations or under-sampling $n < m$: matrix is not invertible $\Rightarrow$ Problem when using M-estimators or Tyler's estimator!

It is an active research on this topic:
see the works of Yuri Abramovich, Olivier Besson, Romain Couillet, Mathew McKay, Ami Wiesel...

Shrinkage Tyler's estimators

Chen estimator

$$\widehat{\Sigma}_C = (1-\beta)\,\frac{m}{n}\sum_{i=1}^{n}\frac{\mathbf{z}_i\mathbf{z}_i^H}{\mathbf{z}_i^H\widehat{\Sigma}_C^{-1}\mathbf{z}_i} + \beta\mathbf{I}$$

subject to the constraint $\mathrm{Tr}(\widehat{\Sigma}) = m$ and for $\beta \in (0,1]$.

- Originally introduced in
  Y. Abramovich and N. K. Spencer, "Diagonally loaded normalised sample matrix inversion (LNSMI) for outlier-resistant adaptive filtering," *in Acoustics, Speech and Signal Processing, IEEE International Conference on, ICASSP-07*, vol. 3, pp. 1105-1108, 2007.
- Existence, uniqueness and algorithm convergence proved in
  Y. Chen, A. Wiesel, and A. O. Hero, "Robust shrinkage estimation of high-dimensional covariance matrices," *Signal Processing, IEEE Transactions on*, vol. 59, no. 9, pp. 4097-4107, 2011.

Shrinkage Tyler's estimators

Pascal estimator

$$\widehat{\Sigma}_P = (1-\beta)\,\frac{m}{n}\sum_{i=1}^{n}\frac{\mathbf{z}_i\mathbf{z}_i^H}{\mathbf{z}_i^H\widehat{\Sigma}_P^{-1}\mathbf{z}_i} + \beta\mathbf{I}$$

subject to the **no** trace constraint but for $\beta \in (\bar{\beta}, 1]$, where $\bar{\beta} := \max(0, 1 - n/m)$.

- Existence, uniqueness and algorithm convergence proved in
  F. Pascal, Y. Chitour, and Y. Quek, "Generalized robust shrinkage estimator and its application to STAP detection problem," *Signal Processing, IEEE Transactions on (submitted to)*, 2014 arXiv:1311.6567.

$\widehat{\Sigma}_P$ (naturally) verifies $\mathrm{Tr}(\widehat{\Sigma}_P^{-1}) = m$ for all $\beta \in (0,1]$

Shrinkage Tyler's estimators

The main challenge is to find the optimal β!

One (theoretical) answer is given thanks to RMT in ...

R. Couillet and M. R. McKay,"Large Dimensional Analysis and Optimization of Robust Shrinkage Covariance Matrix Estimators," arXiv preprint arXiv:1401.4083, 2014.

where it is also proved that

- Both estimators have asymptotically the same performance (achieved for a different value of beta)
- They asymptotically perform as a normalized version of the Ledoit-Wolf estimator.

O. Ledoit and M. Wolf, "A well-conditioned estimator for large-dimensional covariance matrices," *Journal of multivariate analysis* , vol. 88, no. 2, pp. 365-411, 2004.

Outline

Context and difficulties

Problem

Now, the statistical mean is non null $\Rightarrow$ *M*-estimator of the mean is required

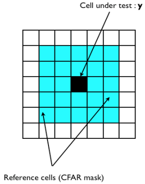
$$\hat{\boldsymbol{\mu}} = \frac{\sum_{i=1}^{n} u_1(t_i)\,\mathbf{z}_i}{\sum_{i=1}^{n} u_1(t_i)} \text{ and } \widehat{\Sigma} = \frac{1}{n}\sum_{i=1}^{n} u_2\left(t_i^2\right)\left(\mathbf{z}_i - \hat{\boldsymbol{\mu}}\right)\left(\mathbf{z}_i - \hat{\boldsymbol{\mu}}\right)^H,$$

where $t_i = \left(\left(\mathbf{z}_i - \hat{\boldsymbol{\mu}}\right)^H \widehat{\Sigma}^{-1}\left(\mathbf{z}_i - \hat{\boldsymbol{\mu}}\right)\right)^{1/2}$ and $u_1(.)$, $u_2(.)$ denote any real-valued *weight functions* (following the conditions of Maronna).

⚠**No proofs of existence, uniqueness, consistency and convergence of the recursive algorithm!**

Methodology

- Rectangular CFAR mask $k \times k$ for different steering vectors $\mathbf{p}$.
- For each $\mathbf{y}$, computation of the detector $\Lambda_{ANMF}(\widehat{\Sigma})$.
- Mask moving all over the hyperspectral image.



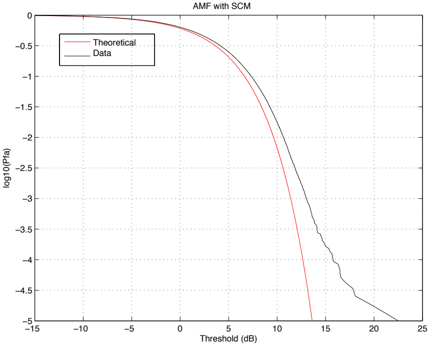
Assumptions

- Pixels of the mask are statistically independent, i.e. spatially independence.
- Pixels of the mask are identically distributed.

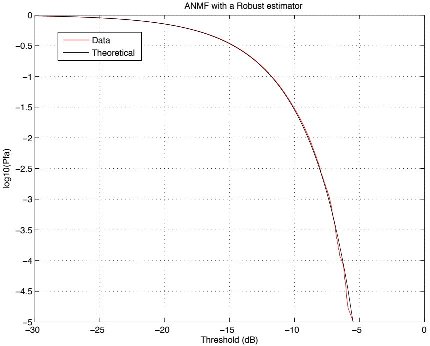
FA regulation proved in non-zero mean Gaussian case

J. Frontera-Pons, F. Pascal, and J. Ovarlez, "False-alarm regulation for target detection in hyperspectral imaging," in *Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP), 2013 IEEE 5th International Workshop on*, pp. 161-164, IEEE, 2013.

False Alarm regulation



(a) AMF-H detector with the SCM



(b) ANMF-H detector with the Tyler's est.

Figure: Probability of false alarm versus the detection threshold for $m = 50$ and $n = 168$

Detection performance

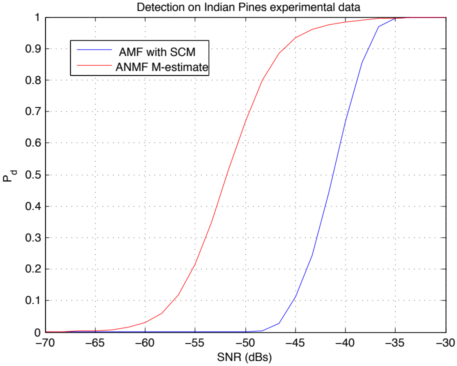


Figure: Detection probability versus SNR for a $P_{fa} = 10^{-2}$

Improvement of $\simeq$ 10 dB in detection due to the detection test and due to the more appropriate covariance matrix estimator

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References and other applications

There have been other applications for CES distributions and robust estimators...

One can cite:

- Multivariate radar imaging

  G. Vasile, J-P. Ovarlez, F. Pascal and C. Tison, "Coherency Matrix Estimation of Heterogeneous Clutter in High-Resolution Polarimetric SAR Images," *Geoscience and Remote Sensing, IEEE Transactions on*, vol. 48, pp. 1809-1826, 2010.

- Image processing

  F. Pascal, L. Bombrun, J.-Y. Tourneret and Y. Berthoumieu, "Parameter Estimation for Multivariate Generalized Gaussian Distributions," *Signal Processing, IEEE Transactions on*, vol. 61, no. 23, pp. 5960-5971, 2013.

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