RMT for Whitening Space Correlation and Applications to Radar Detection

Frederic Pascal[1], Romain Couillet[1], Jean-Philippe Ovarlez[2] and Maria S. Greco[3]

[1] L2S, CentraleSupélec, CNRS, 3 rue Joliot Curie, 91190 Gif-sur-Yvette, France
[2] SONDRA/ONERA, CentraleSupélec, 3 rue Joliot Curie, 91190 Gif-sur-Yvette, France
[3] Dipartimento di Ingegneria dell'Informazione, University of Pisa, Pisa 56122, Italy

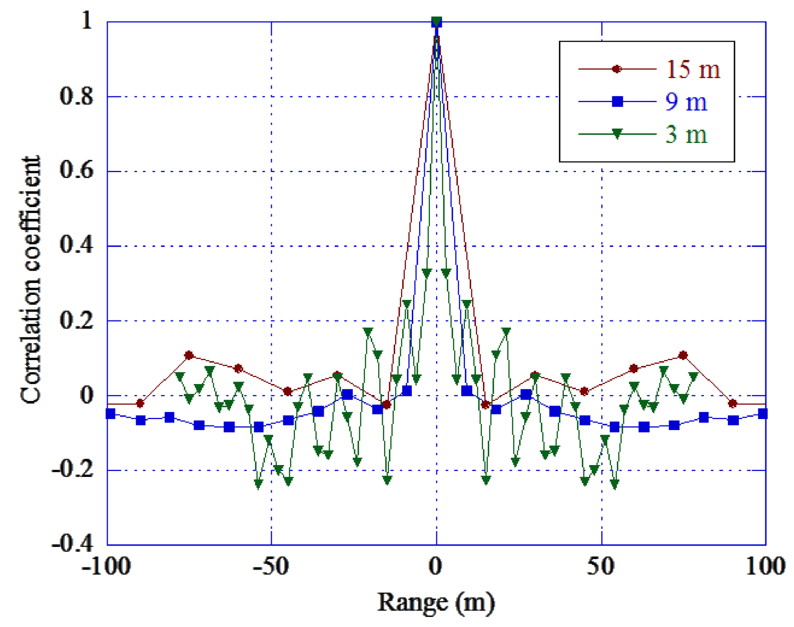
Motivations and Problem Formulation

In many radar and imagery applications, data $\{\mathbf{y}_i\}_{i\in[1,K]}$ can be viewed as a joint spatial and temporal process:

- For high resolution radar, the sea clutter is clearly jointly spatially and temporally correlated,
- In mutichannel (polarimetric, interferometric or multi-temporal) SAR imaging, the multivariate vector characterizing each spatial pixel of the image is correlated over the channels but can also be strongly correlated with those of neighbourhood pixels,
- When the signal is oversampled ($Fe = k\,B$), the sampled noise is correlated and the measurements are not independent anymore.

In the radar community, one generally supposes that the vectors of information collected over a spatial support are **identically and independently distributed**.

The aim of this work is to relax this hypothesis through the use of recent Random Matrix Theory results.



Sea clutter spatial correlation, IPIX radar [M. Greco *et al*, 2006].

The matrix $\mathbf{T}$ is considered Toeplitz, i.e., for all i,j, $\mathbf{T}_{i,j} = t_{|i-j|}$ for $t_0 = 1$ and $t_k \in \mathbb{C}$, and positive definite.

Example: $N = 2$, $K = 3$

$$\mathbf{C} = \underbrace{\underbrace{\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}^{1/2}}_{\text{Temporal correlation}} \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \end{pmatrix}}_{\text{Temporal or Spectral Measurements}} \underbrace{\begin{pmatrix} t_0 & t_1 & t_2 \\ t_1 & t_0 & t_1 \\ t_2 & t_1 & t_0 \end{pmatrix}^{1/2}}_{\text{Spatial correlation}}.$$

Problem

Detection of a complex signal corrupted by an additive Gaussian noise $\mathbf{c} \sim \mathcal{CN}(\mathbf{0}_N, \mathbf{M})$ in a N-dimensional complex observation vector $\mathbf{y}$:

$$\begin{cases} H_0 : \mathbf{y} = \mathbf{c} & \mathbf{y}_i = \mathbf{c}_i \;\; i = 1,\dots,K \\ H_1 : \mathbf{y} = \alpha\,\mathbf{p} + \mathbf{c} & \mathbf{y}_i = \mathbf{c}_i \;\; i = 1,\dots,K \end{cases},$$

where $\mathbf{p}$ is a perfectly known complex steering vector, α is the unknown signal amplitude and where the $\mathbf{c}_i \sim \mathcal{CN}(\mathbf{0}_N, \mathbf{M})$ are K signal-free non independent measurements. The covariance matrix $\mathbf{M}$ characterizes the temporal or spectral correlation within the components of the noise vectors.

To model the spatial dependency between the secondary data, from the Gaussian assumption on $\mathbf{c}_i$, we may write $\mathbf{C} = [\mathbf{c}_1, \dots, \mathbf{c}_K]$ under the following form:

$$\mathbf{C} = \mathbf{M}^{1/2}\,\mathbf{X}\,\mathbf{T}^{1/2},$$

where $\mathbf{M} \in \mathbb{C}^{N\times N}$ and $\mathbf{T} \in \mathbb{C}^{K\times K}$ are both nonnegative definite, $\mathbf{X}$ is standard Gaussian $\mathcal{CN}(\mathbf{0}_N, \mathbf{I}_N)$, and $\mathbf{T}$ satisfies the normalization $\frac{1}{K}\mathrm{tr}\;\mathbf{T} = 1$.

Theoretical Results

Let $\{\mathbf{y}_i\}_{i\in[1,K]} \sim \mathcal{CN}(\mathbf{0_N}, \mathbf{M})$. MLE of $\mathbf{M}$ is given by $\hat{\mathbf{M}} = \frac{1}{K}\sum_{i=1}^{K} \mathbf{y}_i\,\mathbf{y}_i^H$.

Random Matrix Regime

- No longer valid if $N, K \to \infty$ with $N/K \to c \in [0, \infty[$: $\left|\left|\hat{\mathbf{M}} - \mathbf{M}\right|\right| \nrightarrow 0$,
- For practical large N, K with $N \simeq K$, it can lead to dramatically wrong conclusions (even $N = K/100$).

Proposition: Consistent Estimation for T [Couillet *et al*, 2015]

As $N, K \to \infty$ *such that* $N/K \to c \in [0, \infty[$, *and for every* $\beta < 1$,

$$N^{\beta}\left\|\hat{\mathbf{T}} - \left(\frac{1}{N}\mathrm{tr}\;\mathbf{M}\right)\mathbf{T}\right\|_F \xrightarrow{a.s.} 0\,,\; \hat{\mathbf{T}} \propto \mathcal{T}\left[\frac{1}{\mathbf{N}}\mathbf{C}^{\mathbf{H}}\,\mathbf{C}\right]$$

where $\mathcal{T}[\cdot]$ *is the Toeplitzification operator:* $(\mathcal{T}[\mathbf{X}])_{ij} = \frac{1}{K}\sum_{k=1}^{K} \mathbf{X}_{k,k+|i-j|}$.

Let us define a time-whitened SCM of $\mathbf{C}$: $\hat{\mathbf{M}} \triangleq \frac{1}{K}\mathbf{C}\left(\mathcal{T}\left[\frac{1}{N}\mathbf{C}^H\mathbf{C}\right]\right)^{-1}\mathbf{C}^H$.

As a corollary of the previous proposition, with $N, K \to \infty$ with $N/K \to c \in (0, \infty)$, a normalized version of the time-uncorrelated SCM can be so asymptotically recovered:

$$N^{\beta}\left\|\hat{\mathbf{M}} - \frac{\frac{1}{N}\mathbf{M}^{\frac{1}{2}}\mathbf{X}\,\mathbf{X}^H\mathbf{M}^{\frac{1}{2}}}{\frac{1}{N}\mathrm{tr}\;\mathbf{M}}\right\| \xrightarrow{a.s.} 0\,.$$

As a consequence, most functions of $\hat{\mathbf{M}}$ (having fluctuations at a rate lower than $N^{-\beta}$, $\beta < 1$) have the same behavior as those of the traditional time-uncorrelated sample covariance estimator $\frac{1}{N}\mathbf{M}^{\frac{1}{2}}\mathbf{X}\mathbf{X}^H\mathbf{M}^{\frac{1}{2}}$ (inaccessible), up to a constant.

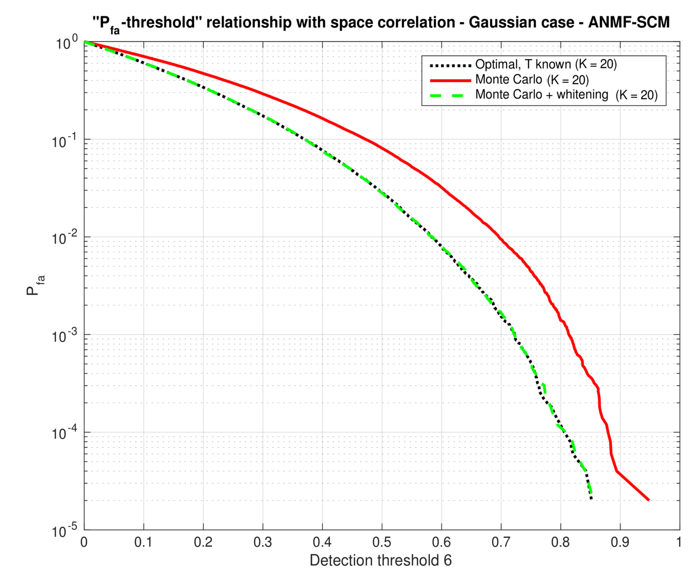
Radar detection with ANMF

$$\Lambda_{ANMF}(\mathbf{y}, \widehat{\mathbf{M}}) = \frac{|\mathbf{p}^H\,\widehat{\mathbf{M}}^{-1}\,\mathbf{y}|^2}{(\mathbf{p}^H\,\widehat{\mathbf{M}}^{-1}\,\mathbf{p})(\mathbf{y}^H\,\widehat{\mathbf{M}}^{-1}\,\mathbf{y})} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

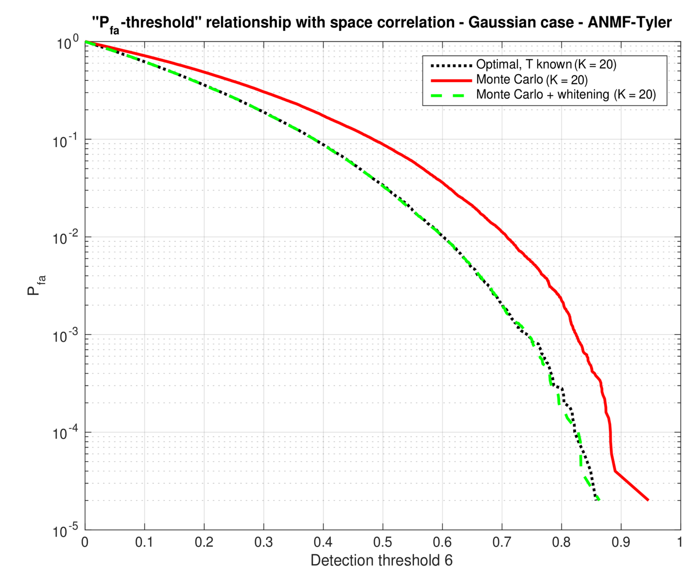
Data: joint spatial and time correlated Gaussian data, $N = 10$, $K = 20$, $\mathbf{M} = \left(\rho_{\mathbf{M}}^{|i-j|}\right)_{i,j\in[1,N]}$, $\mathbf{T} = \left(\rho_{\mathbf{T}}^{|i-j|}\right)_{i,j\in[1,K]}$ with $\rho_{\mathbf{M}} = 0.5$, $\rho_{\mathbf{T}} = 0.9$.

False Alarm regulation

ANMF-SCM

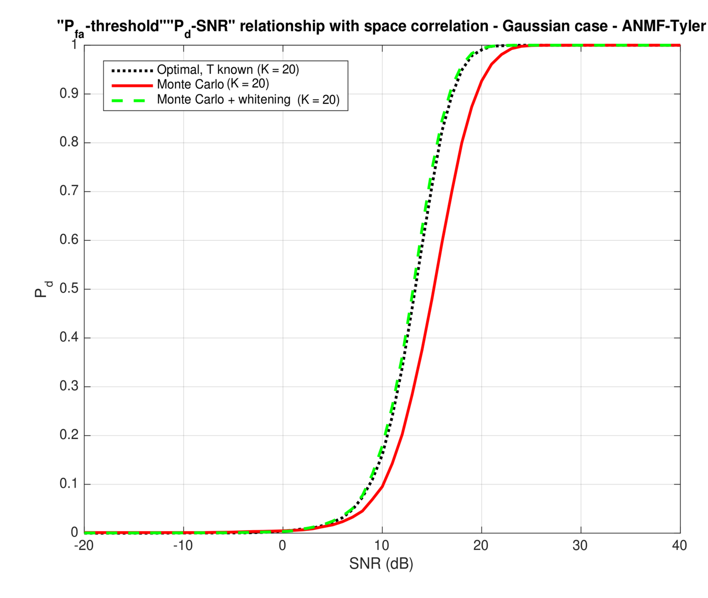


ANMF-Tyler

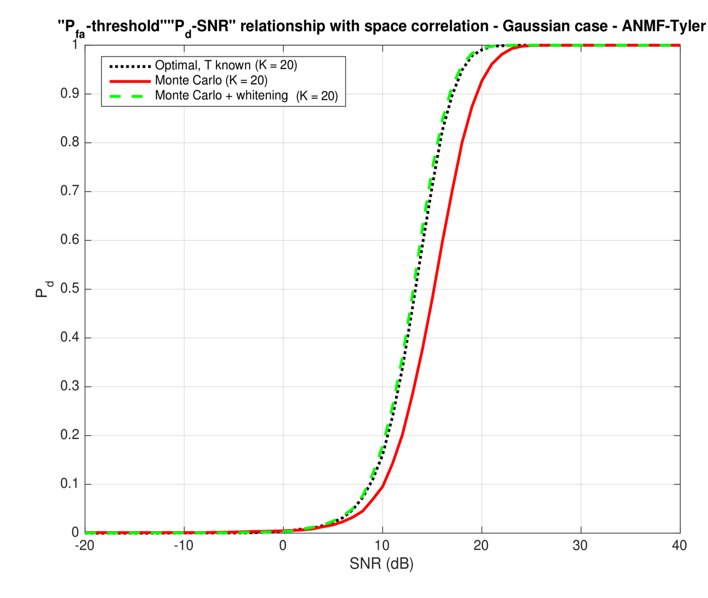


Detection Performance

ANMF-SCM



ANMF-Tyler



Conclusions on joint estimation of spatial and temporal covariance matrices for adaptive radar detection

- This estimation was efficiently performed using latest results coming from RMT with a Toeplitz covariance structure assumption for the spatial covariance matrix
- First results show that the ANMF built with these new estimates has significant higher performances, in term of regulation of false alarm and probability of detection versus SNR, than those of the ANMF built with classical estimates supposing erroneously i.i.d. spatially secondary data
- M-estimators taking into account spatial correlation have similar performance as those of SCM

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