ASYMPTOTIC PROPERTIES OF THE ROBUST ANMF

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GOAL OF THE PAPER

Let us considering the following binary hypotheses test:

$$\begin{cases} H_0: & \mathbf{y} = \mathbf{c}, & \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_N \\ H_1: & \mathbf{y} = \mathbf{c} + \alpha\,\mathbf{p}, & \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_N \end{cases}$$

where $\{\mathbf{c}_i\}_{i\in[1,N]}$ are *N* signal free secondary data, ***p*** a known steering vector and where α is the unknown amplitude of the target to be detected.

The goal of this paper is to **derive** the asymptotic distributions of the robust **Adaptive Normalized Matched Filter** (ANMF) **built with any *M*-estimator** under both H_0 and H_1 hypotheses when the additive noise is modeled by **Complex Elliptically Symmetric (CES)** distributions :

$$H\left(\hat{\mathbf{M}}\right) = \frac{|\mathbf{p}^H \hat{\mathbf{M}}^{-1}\mathbf{y}|^2}{\left(\mathbf{p}^H \hat{\mathbf{M}}^{-1}\mathbf{p}\right)\left(\mathbf{y}^H \hat{\mathbf{M}}^{-1}\mathbf{y}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda.$$

MODEL OF THE BACKGROUND

Complex Elliptically Symmetric Distributions [*Kelker 70, Olilla 03*]**:**

$$f_{\mathbf{c}}(\mathbf{c}) = |\mathbf{M}|^{-1}\, h_m\left(\mathbf{c}^H \mathbf{M}^{-1}\mathbf{c}\right)$$

- $\mathbf{c}$ is a random complex *m*-vector characterizing each pixel of the SAR image,
- $\mathbf{M}$ is the scatter matrix (covariance, up to a scalar factor),
- $h_m(.)$, usually called density generator, is assumed to be known,
- This model takes into account the **spatial heterogeneity** and/or **non-Gaussianity** of the noise (from cell to cell, pixel to pixel) by the texture PDF $p(\tau)$
- The scatter matrix **M** can model the temporal fluctuations structure of the noise (ICM), the correlation between polarimetric/interferometric channels, the correlation existing within the spectral bands, ...

ROBUST ESTIMATION OF THE BACKGROUND PARAMETERS

For an unknown but deterministic texture parameter, the Maximum Likelihood Estimate (MLE) $\hat{\mathbf{M}}_{FP}$ of the Covariance **M**, called the **Fixed Point** or **Tyler**'s *M*-estimator, is the solution of the following implicit equation:

$$\hat{\mathbf{M}}_{FP} = \frac{m}{N}\sum_{k=1}^{N} \frac{\mathbf{c}_k\,\mathbf{c}_k^H}{\mathbf{c}_k^H \hat{\mathbf{M}}_{FP}^{-1}\mathbf{c}_k}$$

The Fixed Point belongs to the family of *M*-estimators (*Robust Statistics [Huber 1964, Maronna 1976, Yohai 2006]*) **in the more general context of CES distributions:**

$$\hat{\mathbf{M}} = \frac{1}{N}\sum_{k=1}^{N} u\left(\mathbf{c}_k^H \hat{\mathbf{M}}^{-1}\mathbf{c}_k\right)\mathbf{c}_k\,\mathbf{c}_k^H$$

- *u(.)* is a weighting function acting on the quadratic form,
- Existence and uniqueness of the solution have been proven provided u(.) satisfy given conditions [*Maronna 1976*],
- Robust to outliers, to the presence of strong targets or high impulsive samples in the reference cells,
- Generalization of MLEs: $u(t) = -h_m'(t)/h_m(t)$

ASYMPTOTIC PROPERTIES OF *M*-ESTIMATORS

For any *M*-estimators and any function H(.), homogeneous of degree 0, i.e. satisfying the property $H(\mathbf{M}) = H(\alpha\,\mathbf{M})$

$$\sqrt{N}\left(\mathrm{vec}(\hat{\mathbf{M}} - \mathbf{M})\right) \xrightarrow{d} \mathcal{GCN}\left(\mathbf{0}_{m^2,1}, \mathbf{\Sigma}_M, \mathbf{\Omega}_M\right),$$

where
$$\begin{aligned} \mathbf{\Sigma}_M &= \nu_1\,\mathbf{M}^T \otimes \mathbf{M} + \nu_2\,\mathrm{vec}(\mathbf{M})\,\mathrm{vec}(\mathbf{M})^H, \\ \mathbf{\Omega}_M &= \nu_1\,(\mathbf{M}^T \otimes \mathbf{M})\,\mathbf{K} + \nu_2\,\mathrm{vec}(\mathbf{M})\,\mathrm{vec}(\mathbf{M})^T \end{aligned}$$

$$\sqrt{N}\left(H\left(\hat{\mathbf{M}}\right) - H(\mathbf{M})\right) \xrightarrow{d} \mathcal{GCN}\left(\mathbf{0}_{r,1}, \mathbf{\Sigma}_H, \mathbf{\Omega}_H\right),$$

where
$$\begin{aligned} \mathbf{\Sigma}_H &= \nu_1\,H'(\mathbf{M})\,(\mathbf{M}^T \otimes \mathbf{M})\,H'(\mathbf{M})^H, \\ \mathbf{\Omega}_H &= \nu_1\,H'(\mathbf{M})\,(\mathbf{M}^T \otimes \mathbf{M})\,\mathbf{K}\,H'(\mathbf{M})^T \\ H'(\mathbf{M}) &= \frac{\partial H(\mathbf{M})}{\partial \mathrm{vec}(\mathbf{M})} \end{aligned}$$

- $\nu_1 = 1, \nu_2 = 0$ for the Sample Covariance Matrix $\hat{\mathbf{M}}_{SCM} = \frac{1}{N}\sum_{k=1}^{N}\mathbf{c}_k\,\mathbf{c}_k^H$
- $\nu_1 = (m+1)/m, \nu_2 = -(m+1)/m^2$ for Tyler's estimator

These important properties mean that:

- any *M*-estimator built with N secondary data (asymptotically) behaves like the SCM in Gaussian environment but with a slight smaller degrees of freedom N/ν_1,
- any function H built with a *M*-estimator behaves like those built with SCM in Gaussian environment but with a slight smaller degrees of freedom N/ν_1.

ASYMPTOTIC PROPERTIES OF ANMF BUILT WITH *M*-ESTIMATORS

When considering the ANMF test which is homogeneous of degree 0,

$$H\left(\hat{\mathbf{M}}\right) = \frac{|\mathbf{p}^H \hat{\mathbf{M}}^{-1}\mathbf{y}|^2}{\left(\mathbf{p}^H \hat{\mathbf{M}}^{-1}\mathbf{p}\right)\left(\mathbf{y}^H \hat{\mathbf{M}}^{-1}\mathbf{y}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda.$$

One has the following result

$$\sqrt{N}\left(H(\hat{\mathbf{M}}) - H(\mathbf{M})\right) \xrightarrow{d} \mathcal{N}\left(0, \mathbf{\Sigma}_H\right),$$

where $\mathbf{\Sigma}_H = \mathbf{\Omega}_H = 2\,\nu_1\,H(\mathbf{M})\,(H(\mathbf{M}) - 1)^2$

Conditioned to the cell under test **y**, one obtains the following asymptotic behavior of the ANMF

$$H\left(\hat{\mathbf{M}}\right) \xrightarrow{d} \mathcal{N}\left(H(\mathbf{M}), \frac{2\,\nu_1}{N}\,H(\mathbf{M})\,(H(\mathbf{M}) - 1)^2\right).$$

- Conditionally to a cell under test **y** containing Gaussian noise

$$p_{H(\mathbf{M})}(u) = e^{-\delta}\,\beta_{1,m-1}(u)\,{}_1F_1(m, 1; u\,\delta) \quad \text{where} \quad \delta = \alpha^2\,\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p}$$

- Conditionally to a cell under test **y** containing SIRV noise

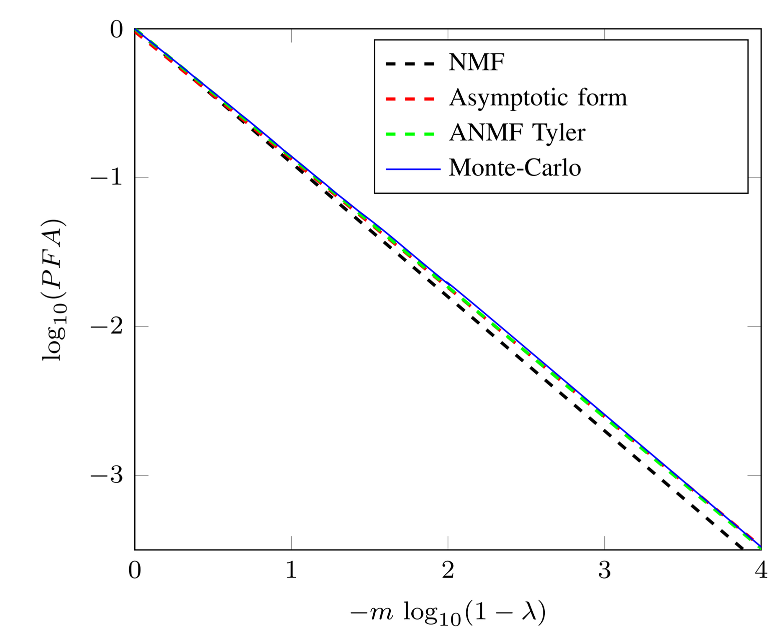
$$p_{H(\mathbf{M})}(u) = \int_0^\infty e^{-\delta/\tau}\,\beta_{1,m-1}(u)\,{}_1F_1\left(m, 1; \frac{u\,\delta}{\tau}\right) p_\tau(\tau)\,d\tau\,.$$

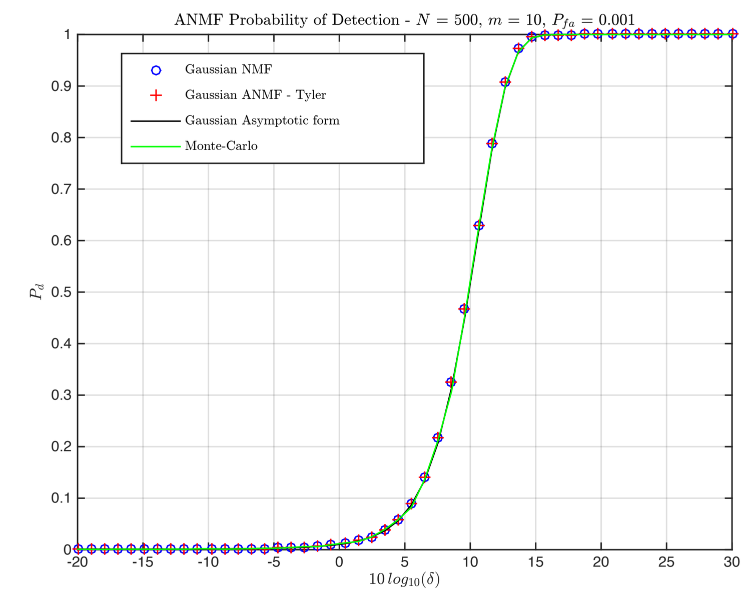
PERFORMANCE ANALYSIS OF ANMF BUILT WITH *M*-ESTIMATORS

- Conditionally to a cell under test y containing Gaussian noise

$$P_{fa} = 1 - \int_0^1 \beta_{1,m-1}(x)\,\Phi\left(\frac{\sqrt{N}\,(\lambda - x)}{\sqrt{2\,\nu_1\,x\,(x-1)^2}}\right) dx\,.$$

$$P_d = 1 - \int_0^1 \beta_{1,m-1}(x)\,e^{\delta\,(x-1)}\,{}_1F_1(1-m, 1; -x\,\delta)\,\Phi\left(\frac{\sqrt{N}\,(\lambda - x)}{\sqrt{2\,\nu_1\,x\,(x-1)^2}}\right) dx\,.$$

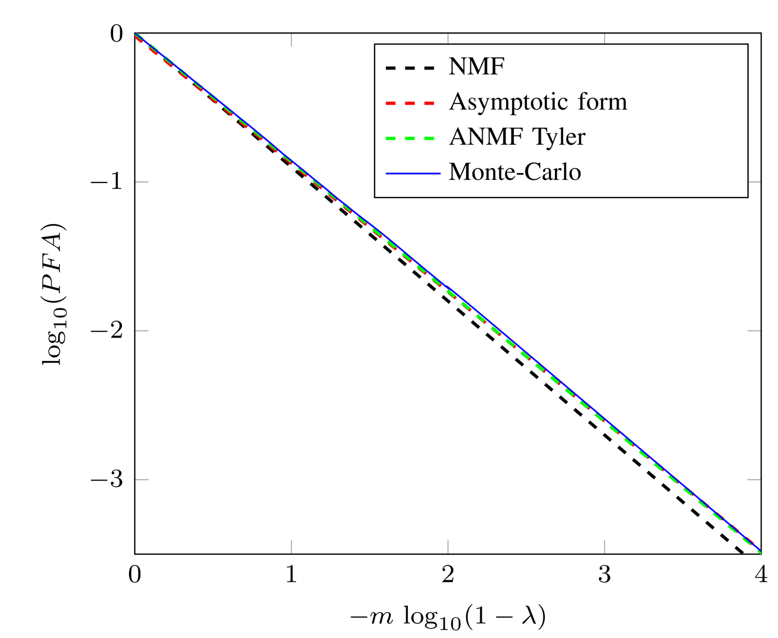


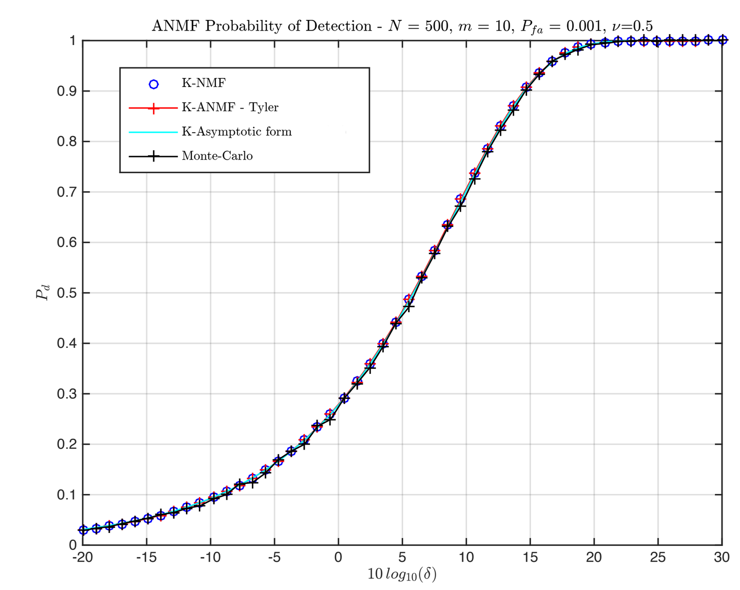


- Conditionally to a cell under test y containing K-distributed noise

$$P_{fa} = 1 - \int_0^1 \beta_{1,m-1}(x)\,\Phi\left(\frac{\sqrt{N}\,(\lambda - x)}{\sqrt{2\,\nu_1\,x\,(x-1)^2}}\right) dx\,.$$

$$P_d = 1 - \int_0^\infty p_\tau(\tau)\,d\tau \int_0^1 \beta_{1,m-1}(x)\,e^{\delta\,(x-1)/\tau} \times {}_1F_1\left(1-m, 1; -x\,\frac{\delta}{\tau}\right)\,\Phi\left(\frac{\sqrt{N}\,(\lambda - x)}{\sqrt{2\,\nu_1\,x\,(x-1)^2}}\right) dx$$





CONCLUSION

- Derivation of the **asymptotic performance of the robust ANMF built with any *M*-estimator**
- Good approximation validated both by Monte-Carlo and by the correction of the degree of freedom in the Gaussian ANMF statistics
- Valid for quite small number of secondary data