Asymptotic Detection Performance of the Robust ANMF

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Motivations

Let us considering the following binary hypotheses test:

$$\begin{cases} H_0 : \mathbf{y} = \mathbf{c}, & \mathbf{y}_i = \mathbf{c}_i, \quad i = 1, \dots, N \\ H_1 : \mathbf{y} = \alpha\,\mathbf{p} + \mathbf{c}, & \mathbf{y}_i = \mathbf{c}_i, \quad i = 1, \dots, N \end{cases},$$

where $\mathbf{c}$ is an additive noise, $\{\mathbf{c}_i\}_{i\in[1,N]}$ are N signal-free secondary data, $\mathbf{p}$ a known steering vector and where α is the unknown amplitude of the target.

In partially homogeneous Gaussian environment (i.e. $\{\mathbf{c}_i\}_{i\in[1,N]} \sim \mathbb{C}\mathcal{N}(\mathbf{0}_m, \mathbf{M})$, $\mathbf{c} \sim \mathbb{C}\mathcal{N}(\mathbf{0}_m, \sigma^2\,\mathbf{M})$) when $\mathbf{M}$ is known and σ^2 unknown, the GLRT is the well known Normalized Matched Filter [L. Scharf]:

$$H(\mathbf{M}) = \frac{|\mathbf{p}^H\mathbf{M}^{-1}\mathbf{y}|^2}{(\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p})(\mathbf{y}^H\mathbf{M}^{-1}\mathbf{y})}.$$

Motivations

When an estimate $\widehat{\mathbf{M}}_{SCM} = \dfrac{1}{N}\displaystyle\sum_{k=1}^{N} \mathbf{c}_k\, \mathbf{c}_k^H$ of $\mathbf{M}$ is plugged into the NMF (two-step GLRT), this results in the so-called ANMF [S. Kraut] whose distribution is given by:

$$p_{H(\widehat{\mathbf{M}}_{SCM})}(x) = \frac{\Gamma(N+1)\, e^{-\delta}}{\Gamma(N-m+1)\,\Gamma(m-1)} \int_0^1 u^{N-m+1}\, \frac{(1-u)^{m-1}\,(1-x)^{N-m}}{(1-u\,x)^{N-m+2}} \times {}_1F_1\left(N-m+2, 1; \frac{\delta\, x\,(1-u)}{1-x\,u}\right)\, du\,. \tag{1}$$

Performances of the NMF and ANMF can be easilly described in terms of:

- Probability of False Alarm P_{fa} versus the detection threshold λ ($\delta = 0$),
- Probability of Detection P_d versus the SNR $\delta = \alpha^2\, \mathbf{p}^H \mathbf{M}^{-1}\, \mathbf{p}/\sigma^2$.

Motivations

Under more severe environment (spikyness, heterogeneity of the background, outliers in secondary data, ...), the performance of ANMF is dramatically degraded. The noise $\mathbf{c}$ and secondary data $\{\mathbf{c}_i\}_{i\in[1,N]}$ cannot be described by conventional Gaussian PDF:

- need to characterize the environment statistics using more general models: Spherically Invariant Random Vectors (SIRV) or Complex Elliptically Symmetric (CES) distributions
- need to propose robust estimators of the background parameters (e.g. covariance matrix): *M*-estimators

The goal of this paper is to derive under both H_0 and H_1 hypotheses the asymptotic distributions (N not too small) of the robust ANMF built with any M-estimator $\widehat{\mathbf{M}}$ when the noise and secondary data are modelled by Complex Elliptically Symmetric (CES) distributions:

$$H(\widehat{\mathbf{M}}) = \frac{\left|\mathbf{p}^H\,\widehat{\mathbf{M}}^{-1}\,\mathbf{y}\right|^2}{\left(\mathbf{p}^H\,\widehat{\mathbf{M}}^{-1}\,\mathbf{p}^H\right)\,\left(\mathbf{y}^H\,\widehat{\mathbf{M}}^{-1}\,\mathbf{y}\right)}.$$

Outline

Modeling the Background

Let $\mathbf{c}$ be a complex circular random vector of length m. $\mathbf{c}$ has a complex elliptically symmetric (CES) distribution ($CE(\boldsymbol{\mu}, \mathbf{M}, h_m)$) if its PDF is

$$g_{\mathbf{c}}(\mathbf{c}) = |\mathbf{M}|^{-1} \, h_m \left((\mathbf{c} - \boldsymbol{\mu})^H \, \mathbf{M}^{-1} \, (\mathbf{c} - \boldsymbol{\mu}) \right),$$

where $h_m : [0, \infty) \rightarrow [0, \infty)$ is the density generator.

- $\boldsymbol{\mu}$ is the statistical mean (generally known or $= \mathbf{0}$)
- $\mathbf{M}$ the scatter matrix

In general (finite second-order moment), $\mathbf{M}$ is equal to the covariance matrix $E[(\mathbf{c} - \boldsymbol{\mu})\,(\mathbf{c} - \boldsymbol{\mu})^H]$ up to a scalar factor.

Attractive clutter modeling

Some important properties

- Large class of distributions: Gaussian, SIRV, MGGD, K-dist., Student-t....
- Closed under affine transformations,
- All sub-vectors of $\mathbf{z}$ have a CES distribution,
- CES stochastic representation theorem: $\mathbf{c} =_d \boldsymbol{\mu} + \tau\, \mathbf{A}\, \mathbf{u}$ where the random scalar texture $\tau \geq 0$ is independent of $\mathbf{u}$ (m-vector uniformly distributed on the sphere) and characterized by its PDF $p_\tau(.)$ and where $\mathbf{M} = \mathbf{A}\, \mathbf{A}^H$,
- SIRV subclass stochastic representation theorem: $\mathbf{c} =_d \boldsymbol{\mu} + \tau\, \mathbf{u}$ where the random scalar texture $\tau \geq 0$ is independent of $\mathbf{u} \sim \mathbb{C}\mathcal{N}(\mathbf{0}_m, \mathbf{M})$ and characterized by its PDF $p_\tau(.)$.

Outline

Estimating the covariance matrix

Let $(\mathbf{c}_1, ..., \mathbf{c}_N)$ be a N-sample $\sim CES(\mathbf{0}_m, \mathbf{M}, h_m)$ (Secondary data).

M-estimator of $\mathbf{M}$

$$\widehat{\mathbf{M}} = \frac{1}{N} \sum_{i=1}^{N} u\left(\mathbf{c}_i^H \,\widehat{\mathbf{M}}^{-1}\, \mathbf{c}_i\right)\, \mathbf{c}_i\, \mathbf{c}_i^H,$$

Maronna (1976), Kent and Tyler (1991)

- Existence
- Uniqueness
- Convergence of the recursive algorithm...

PDF specified $\Rightarrow$ *MLE can be derived:* $u(x) = -h_m'(x)/h_m(x)$
PDF not specified $\Rightarrow$ *general M-estimators are used instead*

Some remarks on the *M*-estimators

- FPE and SCM are "not" (theoretically) *M*-estimators
- FPE is the most robust

FP Estimate (Tyler, 1987; Pascal, 2008)

$$\widehat{\mathbf{M}}_{FPE} = \frac{m}{N} \sum_{i=1}^{N} \frac{\mathbf{c}_i\, \mathbf{c}_i^H}{\mathbf{c}_i^H\, \widehat{\mathbf{M}}_{FPE}^{-1}\, \mathbf{c}_i}$$

- The FPE does not depend on the texture (SIRV or CES distributions)
- Existence, Uniqueness, Convergence of the recursive algorithm...
- True MLE under SIRV noise with unknown deterministic texture $\{\tau_i\}_{i\in[1,N]}$

Asymptotic distribution of complex *M*-estimators

Using the results of Tyler, we derived the following results [Ollila, Mahot, 2013]:

Asymptotic distribution of $\widehat{\mathbf{M}}$

$$\sqrt{N}\ \mathrm{vec}(\widehat{\mathbf{M}} - \mathbf{M}) \xrightarrow{d} \mathbb{C}\mathcal{N}\left(\mathbf{0}_{m^2}, \mathbf{C}, \mathbf{P}\right),$$

where the covariance matrix $\mathbf{C}$ and the pseudo covariance matrix $\mathbf{P}$ are given by:

$$\begin{aligned}\mathbf{C} &= \nu_1 (\mathbf{M}^* \otimes \mathbf{M}) + \nu_2\, \mathrm{vec}(\mathbf{M})\, \mathrm{vec}(\mathbf{M})^H,\\ \mathbf{P} &= \nu_1 (\mathbf{M}^* \otimes \mathbf{M})\, \mathbf{K} + \nu_2\, \mathrm{vec}(\mathbf{M})\, \mathrm{vec}(\mathbf{M})^T,\end{aligned}$$

where $\mathbf{K}$ is the commutation matrix and where the constant ν_1 and ν_2 are completely defined.

Asymptotic distribution of a function of complex M-estimators

- Let $H(\mathbf{V})$ be a r-multivariate function on the set of Hermitian positive-definite matrices, with continuous first partial derivatives and such as $H(\mathbf{V}) = H(\alpha\mathbf{V})$ for all $\alpha > 0$, e.g. the ANMF statistic, the MUSIC statistic.

Asymptotic distribution of $H(\mathbf{M})$

$$\sqrt{N}\left(H(\widehat{\mathbf{M}}) - H(\mathbf{M})\right) \xrightarrow{d} \mathbb{C}\mathcal{N}\left(\mathbf{0}_r, \mathbf{C}_H, \mathbf{P}_H\right)$$

where $\mathbf{C}_H$ and $\mathbf{P}_H$ are defined as

$$\mathbf{C}_H = \nu_1\, H'(\mathbf{M})\left(\mathbf{M}^T \otimes \mathbf{M}\right) H'(\mathbf{M})^H,$$
$$\mathbf{P}_H = \nu_1\, H'(\mathbf{M})\left(\mathbf{M}^T \otimes \mathbf{M}\right) \mathbf{K}_{m,m}\, H'(\mathbf{M})^T,$$

where $H'(\mathbf{M}) = \left(\dfrac{\partial H(\mathbf{M})}{\partial \text{vec}(\mathbf{M})}\right)$.

Two important properties of complex *M*-estimators

- SCM and *M*-estimators share the same asymptotic distribution
- $H(SCM)$ and $H(M\text{-estimators})$ share the same asymptotic distribution (differs from ν_1)

	SCM	*M*-estimators	FP
ν_1	1	ν_1	$(m+1)/m$
ν_2	0	ν_2	$-(m+1)/m^2$

This important result shows that asymptotically and under Gaussian environment:

- any *M*-estimator built with N observations behaves like the SCM but with a slight smaller degree of freedom N/ν_1,
- any function H built with *M*-estimator behaves like those built with SCM but with a slight smaller degree of freedom N/ν_1.

Outline

A particular choice of $H(.)$: the two-step GLRT ANMF

Robust ANMF test (ACE, GLRT-LQ) [Conte, 1995]

$$H(\widehat{\mathbf{M}}) = \Lambda_{ANMF}(\mathbf{y}, \widehat{\mathbf{M}}) = \frac{|\mathbf{p}^H \widehat{\mathbf{M}}^{-1} \mathbf{y}|^2}{(\mathbf{p}^H \widehat{\mathbf{M}}^{-1} \mathbf{p})(\mathbf{y}^H \widehat{\mathbf{M}}^{-1} \mathbf{y})} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

where $\widehat{\mathbf{M}}$ stands for any M-estimators.

- The ANMF is scale-invariant (homogeneous of degree 0), i.e.

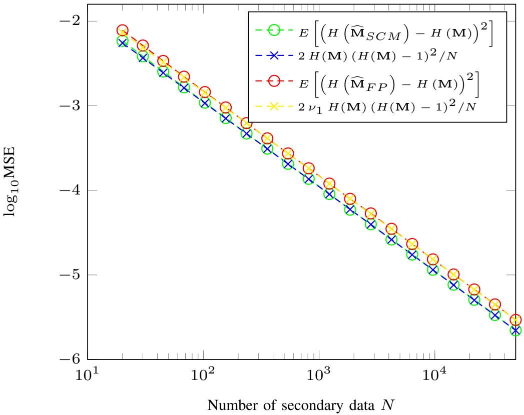
$$\forall \alpha, \beta \in \mathbb{R}, \Lambda_{ANMF}(\alpha\, \mathbf{y}, \beta\, \widehat{\mathbf{M}}) = \Lambda_{ANMF}(\mathbf{y}, \widehat{\mathbf{M}})$$

- The ANMF test is CFAR w.r.t the covariance/scatter matrix $\mathbf{M}$,
- The ANMF test is CFAR w.r.t the texture (SIRV or CES distributions)

Two different ways to derive robust ANMF asymptotic performance

- by correcting the degree of freedom of the M-estimator ($N \to N/\nu_1$) in $p_{H(\widehat{\mathbf{M}}_{SCM})}$ presented in (1) and by conditioning on the texture PDF,
- by exploiting directly the asymptotic distribution of the ANMF [Pascal -Ovarlez 2015]:

$$H(\widehat{\mathbf{M}}) \xrightarrow{d} \mathbb{C}\mathcal{N}\left(H(\mathbf{M}), 2\,\frac{\nu_1}{N}\,H(\mathbf{M})\,\left(H(\mathbf{M})-1\right)^2\right).$$



Empirical variance of the ANMF built with the SCM ($\nu_1 = 1$) and Tyler's M-estimator ($\nu_1 = (m+1)/m$) in Gaussian environment and theoretical asymptotic variance for $m = 3$ and $\mathbf{M} = \mathbf{I}_3$.

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Correcting the degree of freedom of the M-estimator

Evaluation of the performance for a cell under test containing SIRV noise and for any CES secondary data.

$$P_{fa} = \mathbb{P}\left(H(\widehat{\mathbf{M}}) \geq \lambda | H_0\right), \qquad P_d = \mathbb{P}\left(H(\widehat{\mathbf{M}}) \geq \lambda | H_1\right)$$

$$P_{fa} = (1-\lambda)^{N/\nu_1 - m + 1} \, {}_2F_1\left(N/\nu_1 - m + 2, N/\nu_1 - m + 1; N/\nu_1 + 1; \lambda\right),$$

$$P_d = 1 - \int_0^{+\infty} d\tau \int_0^1 du \int_0^{\lambda} u^{N/\nu_1 - m + 1} \, \frac{(1-u)^{m-1}\,(1-x)^{N/\nu_1 - m}}{(1-u\,x)^{N/\nu_1 - m + 2}} \, e^{-\delta/\tau}$$
$$\times \frac{\Gamma(N/\nu_1 + 1)}{\Gamma(N/\nu_1 - m + 1)\,\Gamma(m-1)} \, {}_1F_1\left(N/\nu_1 - m + 2, 1; \frac{\delta}{\tau}\,\frac{x\,(1-u)}{1-x\,u}\right) \, p_\tau(\tau)\, dx$$

Outline

Exploiting the asymptotic distribution of the robust ANMF

$$H(\widehat{\mathbf{M}}) \xrightarrow{d} \mathbb{C}\mathcal{N}\left(H(\mathbf{M}), 2\,\frac{\nu_1}{N}\,H(\mathbf{M})\,\left(H(\mathbf{M})-1\right)^2\right)$$

The distribution of $H(\widehat{\mathbf{M}})$ is conditioned to the distribution $p_{H(\mathbf{M})}$ of $H(\mathbf{M})$:

- the cell under test contains Gaussian noise:

$$p_{H(\mathbf{M})}(u) = e^{-\delta}\,\beta_{1,m-1}(u)\,{}_1F_1\left(m, 1; u\,\delta\right).$$

- the cell under test contains SIRV noise:

$$p_{H(\mathbf{M})}(u) = \int_0^{\infty} e^{-\delta/\tau}\,\beta_{1,m-1}(u)\,{}_1F_1\left(m, 1; \frac{u\,\delta}{\tau}\right)\,p_\tau(\tau)\,d\tau\,.$$

- the cell under test contains general CES noise: No closed-form under H_1. The same as SIRV noise under H_0.

Exploiting the asymptotic distribution of the robust ANMF

Evaluation of the performance for a cell under test containing Gaussian noise and for any CES secondary data.

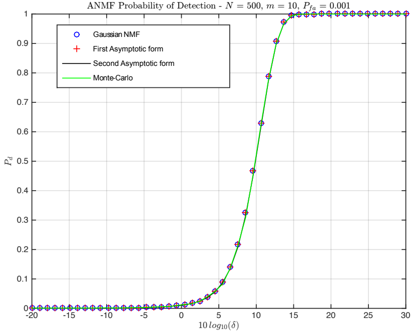
$$P_{fa} = \mathbb{P}\left(H(\widehat{\mathbf{M}}) \geq \lambda | H_0\right), \qquad P_d = \mathbb{P}\left(H(\widehat{\mathbf{M}}) \geq \lambda | H_1\right)$$

$$P_{fa} = 1 - \int_0^1 \beta_{1,m-1}(x)\, \Phi\left(\frac{\sqrt{N}\,(\lambda - x)}{\sqrt{2\,\nu_1\, x\,(x-1)^2}}\right) dx\,.$$

$$P_d = 1 - \int_0^1 \beta_{1,m-1}(x)\, e^{\delta\,(x-1)}\, {}_1F_1\left(1-m, 1; -x\,\delta\right)\, \Phi\left(\frac{\sqrt{N}\,(\lambda - x)}{\sqrt{2\,\nu_1\, x\,(x-1)^2}}\right) dx$$

where $\Phi(.)$ is the cumulative distribution of the Normal distribution.

Ex: K-distributed secondary data and Gaussian noise in the cell under test.



Comparison between P_d and SNR δ relationships for the ANMF built with Tyler's estimator, $m = 10$, $N = 500$ and $P_{fa} = 10^{-3}$, $\mathbf{p} = [1, \ldots, 1]^T$, $\{\mathbf{y}_i\}_{i \in [1,N]} \sim K_\nu$ where K_ν is a K-distribution with shape $\nu = 0.1$. and $\mathbf{y} \sim \mathcal{CN}(\alpha\,\mathbf{p}, \mathbf{M})$ where $\mathbf{M} = \left(\rho^{|i-j|}\right)_{i,j}$ with $\rho = 0.5$

Exploiting the asymptotic distribution of the robust ANMF

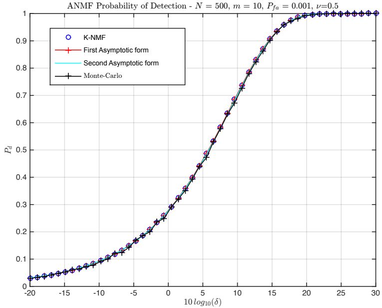
Evaluation of the performance for a cell under test containing SIRV noise with texture $\sim p_\tau(.)$ and for any CES secondary data.

$$P_{fa} = \mathbb{P}\left(H(\widehat{\mathbf{M}}) \geq \lambda | H_0\right), \qquad P_d = \mathbb{P}\left(H(\widehat{\mathbf{M}}) \geq \lambda | H_1\right)$$

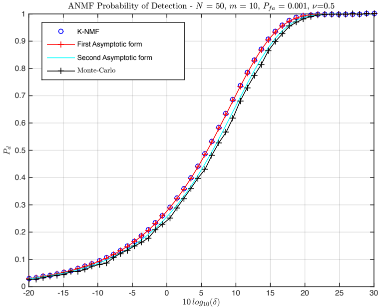
$$P_{fa} = 1 - \int_0^1 \beta_{1,m-1}(x)\,\Phi\left(\frac{\sqrt{N}\,(\lambda - x)}{\sqrt{2\,\nu_1\,x\,(x-1)^2}}\right)\,dx\,.$$

$$P_d = 1 - \int_0^\infty p_\tau(\tau)\,d\tau \int_0^1 \beta_{1,m-1}(x)\,e^{\delta\,(x-1)/\tau}\,{}_1F_1\left(1-m, 1; -x\,\frac{\delta}{\tau}\right) \times \Phi\left(\frac{\sqrt{N}\,(\lambda - x)}{\sqrt{2\,\nu_1\,x\,(x-1)^2}}\right)\,dx$$

where $\Phi(.)$ is the cumulative distribution of the Normal distribution.



$N = 500$



$N = 50$

Comparison between P_d and SNR δ relationships for the NMF, the ANMF built with Tyler's estimator and its asymptotic form, $m = 10$, $\nu_1 = 1.1$ and $P_{fa} = 10^{-3}$, $\mathbf{p} = [1, \ldots, 1]^T$, $\mathbf{y} = \alpha\,\mathbf{p} + \mathbf{c}$ where $\mathbf{c} \sim K_\nu$ where K_ν is a multivariate K-distribution with shape parameter $\nu = 0.5$ and covariance matrix $\mathbf{M} = \left(\rho^{|i-j|}\right)_{i,j}$ with $\rho = 0.5$.

Outline

Conclusions

Two asymptotic approximations of the corresponding robust ANMF distribution have been derived following different approaches:

- First asymptotic distribution is based on the correction of the degrees of freedom of M-estimators
- Second asymptotic distribution is based on the direct exploitation of the asymptotic distribution of the ANMF

These results provide a very good approximation of the robust ANMF distribution in CES environment even for a small number of observations and have been applied to theoretically regulate the false alarm probability and to evaluate the detection performance.