On the Recursions of Robust COMET Algorithm for Convexly Structured Shape Matrix

B. Mériaux[1], C. Ren[1], A. Breloy[2], M.N. El Korso[2], P. Forster[3] and J.-P. Ovarlez[1,4]

[1]SONDRA, CentraleSupélec, France [2]LEME, Paris-Nanterre University, France
[3]SATIE, Paris-Nanterre University, France [4]DEMR/ONERA, Paris-Saclay University, France

Context and Objectives

Shape Matrix:
- normalized version of the covariance matrix
- plays a central role in adaptive signal processing
- often exhibits a particular structure (e.g., Toeplitz for ULAs or AR processes)

Robust framework:
- impulsive noise, heterogeneous environment, high resolution systems
- non-Gaussian distributions $\rightsquigarrow$ outliers rejection

Structured estimation:
- improves the estimation accuracy
- robust framework: RCOMET [1], COCA [2], ...

Purposes: * introduce a recursive version of RCOMET and conduct asymptotic performance analysis
* analyze the recursion convergence for the Hermitian persymmetric structure

Statistical framework and data model

Complex Elliptical Symmetric distribution [3]: $\mathbf{x} \overset{d}{=} \sqrt{Q}\mathbf{A}\mathbf{u} \sim \mathbb{C}\mathcal{ES}_m\left(\mathbf{0}, \mathbf{M} \triangleq \mathbf{A}\mathbf{A}^H, g\right)$

$\mathbf{u} \sim \mathcal{U}_q\left(\mathbb{CS}^q\right)$, random variable $Q \geq 0$ such that $\mathbf{u} \perp\!\!\!\perp Q$ ↲ unknown in practice ↲

Complex Angular Elliptical distribution [2]: $\mathbf{y} = \frac{\mathbf{x}}{\|\mathbf{x}\|} \sim \mathcal{U}_m\left(\mathbf{M}\right), \mathbf{x} \neq \mathbf{0}$

- free from density generator function
- probability density function: $p_{\mathbf{Y}}(\mathbf{y};\mathbf{M}) \propto |\mathbf{M}|^{-1}\left(\mathbf{y}^H\mathbf{M}^{-1}\mathbf{y}\right)^{-m}$
- shape matrix $\mathbf{M}$ defined up to a scale factor $\rightsquigarrow$ needed constraint, e.g., $\mathrm{Tr}\left(\mathbf{M}\right) = m$

Tyler's M-estimator [4]: $\widehat{\mathbf{M}}_{\mathrm{FP}} = \frac{1}{N}\sum_{n=1}^{N}\frac{\mathbf{y}_n\mathbf{y}_n^H}{\mathbf{y}_n^H\widehat{\mathbf{M}}_{\mathrm{FP}}^{-1}\mathbf{y}_n} \triangleq \mathcal{H}\left(\widehat{\mathbf{M}}_{\mathrm{FP}}\right)$ with $\mathbf{y}_n \sim \mathcal{U}_m\left(\mathbf{M}\right)$ i.i.d.

- existence for $N > m$, uniqueness up to a scale factor $\rightsquigarrow$ no ambiguity with $\mathrm{Tr}\left(\widehat{\mathbf{M}}_{\mathrm{FP}}\right) = m$
- Maximum Likelihood estimator of the shape matrix
- convergence of the iterative algorithm $\mathbf{M}_{k+1} = \mathcal{H}\left(\mathbf{M}_k\right)$ towards $\widehat{\mathbf{M}}_{\mathrm{FP}}$ for any $\mathbf{M}_0$
- statistical performance: consistency, unbiasedness, asymptotic Gaussian distribution [5]

Problem setup and RCOMET algorithm

Let be $\mathbf{y}_n \sim \mathcal{U}_m\left(\mathbf{M}_{\mathrm{e}} \triangleq \boldsymbol{\mathcal{M}}\left(\boldsymbol{\mu}_{\mathrm{e}}\right)\right)$, $n = 1, \ldots, N$, i.i.d. such that $N > m$.

We assume:
- $\mathbf{M}_{\mathrm{e}} \in \mathcal{S}$ convex subset of Hermitian positive-definite matrices
- there exists a one-to-one differentiable mapping $\boldsymbol{\mu} \mapsto \boldsymbol{\mathcal{M}}\left(\boldsymbol{\mu}\right)$ from $\mathbb{R}^P$ to $\mathcal{S}$

Unknown parameter: $\boldsymbol{\mu} \in \mathbb{R}^P$ giving a structured estimate $\boldsymbol{\mathcal{M}}\left(\widehat{\boldsymbol{\mu}}\right)$ and with exact value $\boldsymbol{\mu}_{\mathrm{e}}$

RCOMET algorithm [1]: 2-step procedure

- $\widehat{\boldsymbol{\mu}}_0 = \arg\min_{\alpha>0, \boldsymbol{\mu}} \mathrm{Tr}\left[\left\{\left(\widehat{\mathbf{M}}_{\mathrm{FP}} - \alpha\boldsymbol{\mathcal{M}}\left(\boldsymbol{\mu}\right)\right)\widehat{\mathbf{M}}_{\mathrm{FP}}^{-1}\right\}^2\right]$ such that $\mathrm{Tr}\left[\boldsymbol{\mathcal{M}}\left(\widehat{\boldsymbol{\mu}}_0\right)\right] = m$.
- Asymptotic performance: $\widehat{\boldsymbol{\mu}}_0 \overset{\mathcal{P}}{\to} \boldsymbol{\mu}_{\mathrm{e}}$ and $\sqrt{N}\left(\widehat{\boldsymbol{\mu}}_0 - \boldsymbol{\mu}_{\mathrm{e}}\right) \overset{\mathcal{L}}{\to} \mathcal{N}\left(\mathbf{0}, \left[\mathbf{F}\left(\boldsymbol{\mu}_{\mathrm{e}}\right)\right]^{-1}\right)$
  Fisher Information matrix ↲
- Substantial sample support to reach its asymptotic regime
- $\widehat{\mathbf{M}}_{\mathrm{FP}}$ plays both the role of a target together with a metric specification through $\widehat{\mathbf{M}}_{\mathrm{FP}}^{-1}$ $\rightsquigarrow$ splitting these roles leads to a recursive formulation

Recursive RCOMET Procedure

Algorithm: let be $1 \leq K < \infty$ and $\widehat{\boldsymbol{\mu}}_0$ the RCOMET estimate

$$\text{For } k \in [\![1, K]\!] \qquad \widehat{\boldsymbol{\mu}}_k = \arg\min_{\alpha>0, \boldsymbol{\mu}} \mathrm{Tr}\left[\left\{\left(\widehat{\mathbf{M}}_{\mathrm{FP}} - \alpha\boldsymbol{\mathcal{M}}\left(\boldsymbol{\mu}\right)\right)\boldsymbol{\mathcal{M}}\left(\widehat{\boldsymbol{\mu}}_{k-1}\right)^{-1}\right\}^2\right] \qquad (1)$$

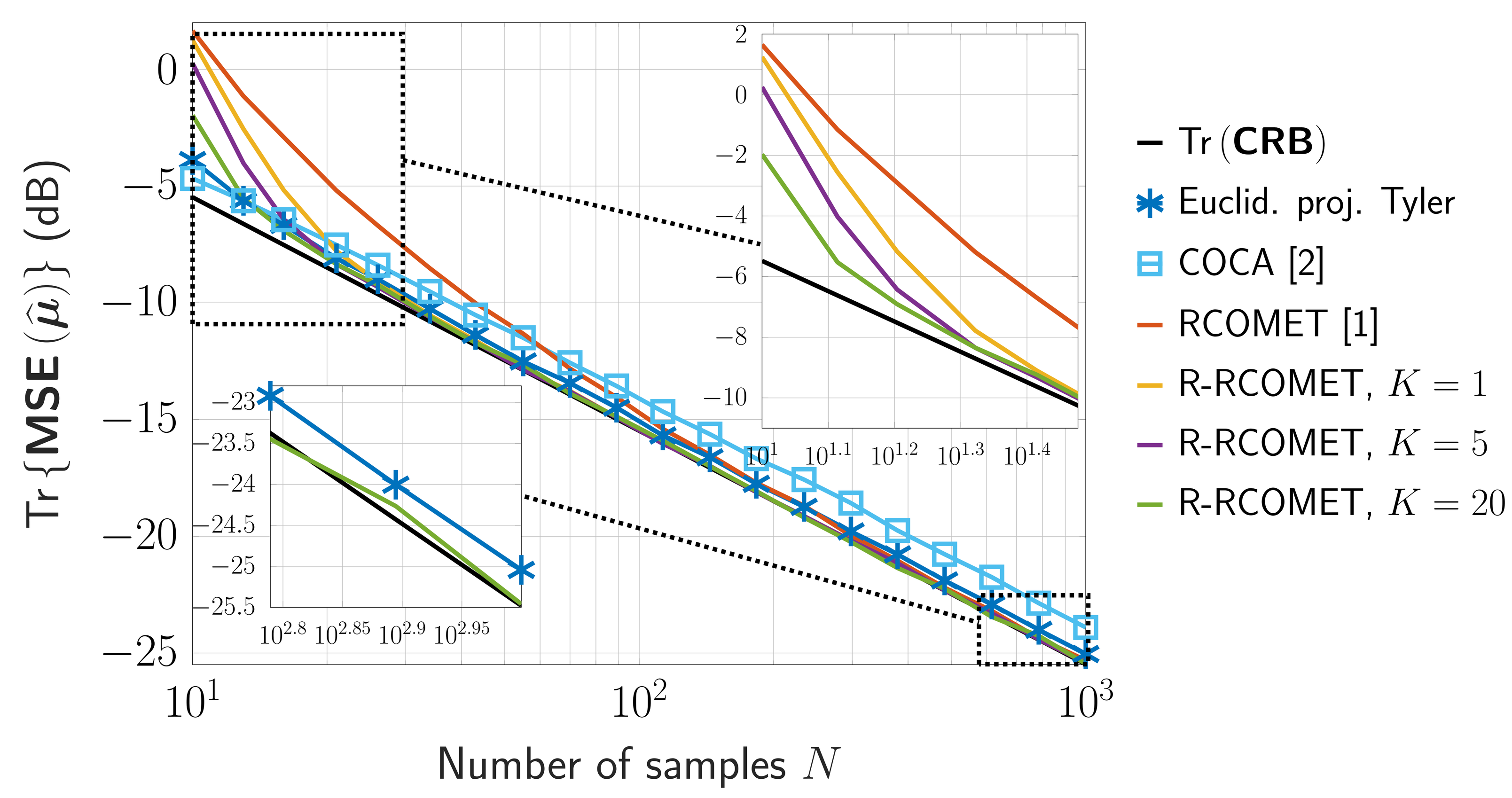
R-RCOMET estimate: $\widehat{\boldsymbol{\mu}}_K$ such that $\mathrm{Tr}\left[\boldsymbol{\mathcal{M}}\left(\widehat{\boldsymbol{\mu}}_K\right)\right] = m$

Asymptotic performance: $\widehat{\boldsymbol{\mu}}_K \overset{\mathcal{P}}{\to} \boldsymbol{\mu}_{\mathrm{e}}$ and $\sqrt{N}\left(\widehat{\boldsymbol{\mu}}_K - \boldsymbol{\mu}_{\mathrm{e}}\right) \overset{\mathcal{L}}{\to} \mathcal{N}\left(\mathbf{0}, \left[\mathbf{F}\left(\boldsymbol{\mu}_{\mathrm{e}}\right)\right]^{-1}\right)$

Comments: ✓ same asymptotic performance as RCOMET
✓ empirical improvement of low sample support performance in most cases
✗ convergence for $K \to \infty$ difficult to conduct in general $\rightsquigarrow$ case by case study

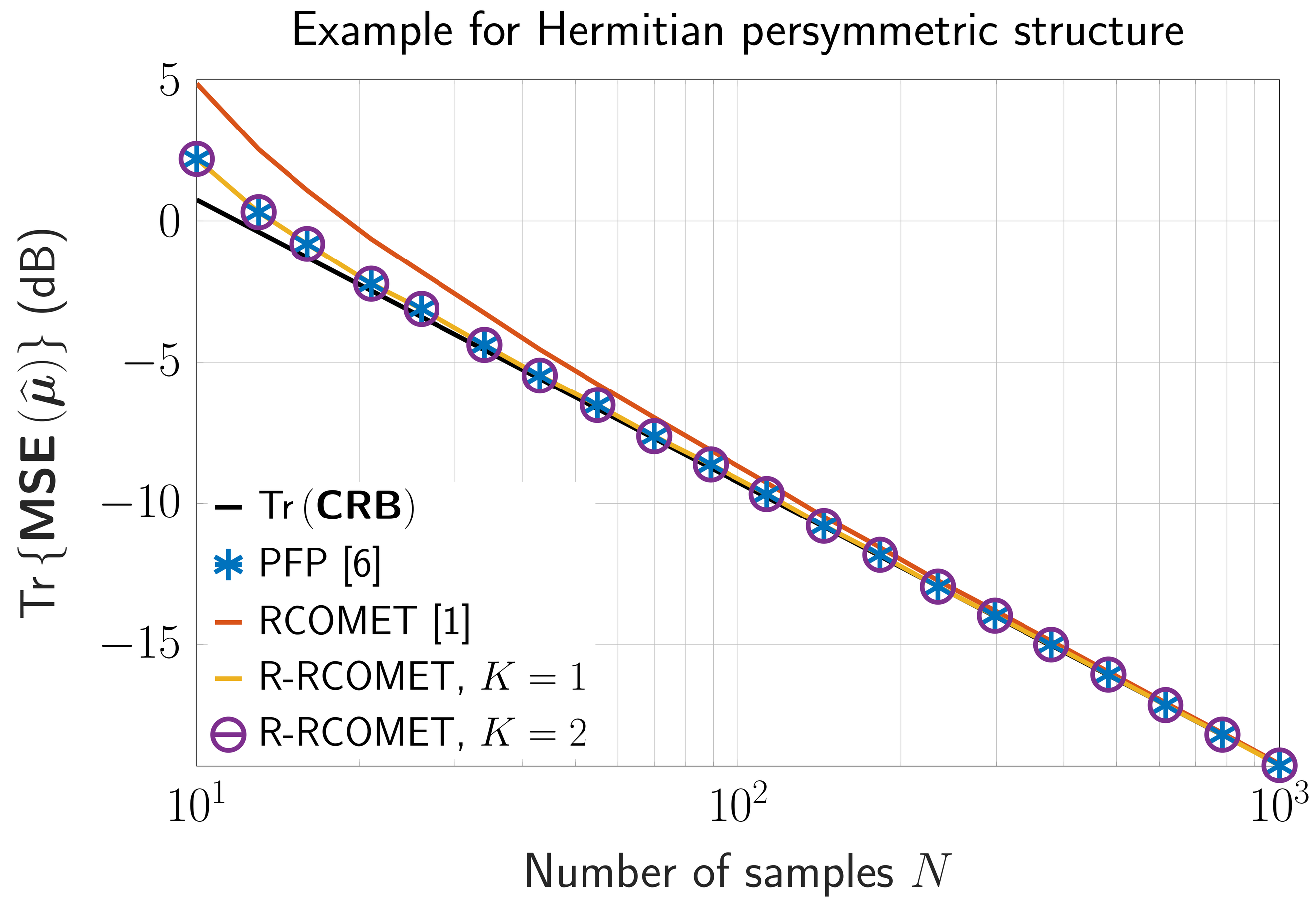
Practical stopping rule: combination of $k \leq K_{\max}$ and $\|\widehat{\boldsymbol{\mu}}_{k+1} - \widehat{\boldsymbol{\mu}}_k\| \leq \varepsilon_{\mathrm{tol}}\|\widehat{\boldsymbol{\mu}}_k\|$.

Example for Hermitian Toeplitz structure



R-RCOMET allows for an interesting performance-computational cost trade-off

Example for Hermitian persymmetric structure



Case: Hermitian persymmetric structure

Hermitian persymmetric matrices set:

$$\mathcal{HP}_m = \left\{\mathbf{A} \in \mathbb{C}^{m\times m} \middle| \mathbf{A} = \mathbf{A}^H \text{ and } \mathbf{A} = \mathbf{J}_m\mathbf{A}^T\mathbf{J}_m\right\}$$

exchange matrix ↲

Natural parameterization:

$\hookrightarrow$ real and imaginary parts of $M_{r,s}$ satisfying $s \geq r$ and $s \leq m + 1 - r$

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2^* & a_5 & a_6 & a_3 \\ a_3^* & a_6^* & a_5 & a_2 \\ a_4^* & a_3^* & a_2^* & a_1 \end{pmatrix} \rightsquigarrow \boldsymbol{\mu} = \begin{pmatrix} a_1 \\ \vdots \\ \vdots \end{pmatrix} \in \mathbb{R}^P$$

with $P = m(m+1)/2$

$\exists\, \boldsymbol{\mathcal{J}} \in \mathbb{R}^{m^2\times P}$ full-rank column, such that $\boldsymbol{\eta}\left(\boldsymbol{\mu}\right) \triangleq \mathrm{vec}\left(\boldsymbol{\mathcal{M}}\left(\boldsymbol{\mu}\right)\right) = \boldsymbol{\mathcal{J}}\boldsymbol{\mu}$

Surprising results for this particular case

Let $\widehat{\boldsymbol{\mu}}_K$ be the R-RCOMET estimate of $\boldsymbol{\mu}_{\mathrm{e}}$. Then,

$\widehat{\boldsymbol{\mu}}_K = \boldsymbol{\mathcal{J}}^{\dagger}\widehat{\boldsymbol{\eta}}_{\mathrm{FP}}\left(\neq \widehat{\boldsymbol{\mu}}_0\right) \quad \forall K \geq 1$ $\rightsquigarrow$ R-RCOMET converges in only one step

$= \arg\min_{\alpha>0, \boldsymbol{\mu}} \|\widehat{\boldsymbol{\eta}}_{\mathrm{FP}} - \alpha\boldsymbol{\mathcal{J}}\boldsymbol{\mu}\|_2^2$ s.t. $\mathrm{Tr}\left[\boldsymbol{\mathcal{M}}\left(\boldsymbol{\mu}\right)\right] = m$ $\rightsquigarrow$ Euclidean projection of $\widehat{\mathbf{M}}_{\mathrm{FP}}$ onto $\mathcal{HP}_m$

$= \boldsymbol{\mathcal{J}}^{\dagger}\left(\mathbf{T}^T \otimes \mathbf{T}^H\right)\mathrm{vec}\left(\frac{1}{2}\left[\mathbf{T}\widehat{\mathbf{M}}_{\mathrm{FP}}\mathbf{T}^H + \mathbf{T}^*\widehat{\mathbf{M}}_{\mathrm{FP}}^T\mathbf{T}^T\right]\right)$ $\rightsquigarrow$ Persymmetric Fixed-Point estimate from [6]

with $\boldsymbol{\mathcal{J}}^{\dagger}$ left inverse of $\boldsymbol{\mathcal{J}}$, $\widehat{\boldsymbol{\eta}}_{\mathrm{FP}} = \mathrm{vec}\left(\widehat{\mathbf{M}}_{\mathrm{FP}}\right)$ and $\mathbf{T}$ unitary matrix, defined in [6, Proposition 1]

Perspectives

- Justification of the low sample support performance improvement
- R-RCOMET convergence study for other structures
- Investigate links with estimators on Riemannian manifolds

References

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