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DISCRETE MELLIN TRANSFORM FOR SIGNAL ANALYSIS

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Abstract. Theoretical wide-band studies generally provide expressions involving stretched forms of the signal. This feature complicates the implementation of the results and suggests the use of a Mellin transform in order to process dilations efficiently. The purpose of the work is to give a tool for practical developments of this idea.

In a first step the definition, properties and time-frequency interpretation of the relevant Mellin transform are given. Then the discretization is developed, leading to a form which can run with any FFT routine. Finally the advantage of the technique is illustrated by computing broad-band radar ambiguity functions and affine time-frequency representations.

1. Time-frequency interpretation of the Mellin transform.

Introduction of the Mellin transform in signal analysis corresponds generally to the search for scale invariant properties [1][2]. The point of view here is quite different since the transformation is only considered as an interesting technique for computation of functionals containing dilations. The main progress is due to the interpretation of the Mellin variable in the time-frequency half-plane which leads to a clear formulation of the sampling problem.

The Mellin transform we consider is defined on the analytic signal S(f) by the relation:

$$\mathcal{M}^{\xi}(\beta) = \int_{\mathbb{R}^+} S(f)\; e^{2i\pi\xi f}\; f^{r+2i\pi\beta}\, df \qquad (1)$$

and by its reciprocal form:

$$S(f) = \int_{\mathbb{R}} \mathcal{M}^{\xi}(\beta)\; e^{-2i\pi\xi f} f^{-2i\pi\beta-r-1} d\beta \qquad (2)$$

The parameter $r\in\mathbb{R}$ corresponds to a physical scaling factor and is left free.
The transformation is unitary, i.e.

$$(S_1, S_2) = \int_{\mathbb{R}} \mathcal{M}_1^{\xi}(\beta)\; \mathcal{M}_2^{\xi *}(\beta)\; d\beta \qquad (3)$$

for the scalar product:

$$(S_1, S_2) = \int_{\mathbb{R}^+} S_1(f)\; S_2^*(f)\; f^{2r+1}\, df \qquad (4)$$

The main interest of the transformation (1)-(2) appears when considering the operation:

$$S(f) \longrightarrow S'(f) = a^{r+1} e^{-2i\pi\xi(1-a)f} S(af) \qquad (5)$$

which corresponds to a dilation of coefficient $a>0$ about a fixed time $\xi\in\mathbb{R}$. In Mellin space transformation (5) is simply expressed by:

$$\mathcal{M}^{\xi}\,[S'] = a^{-2i\pi\beta}\, \mathcal{M}^{\xi}\,[S] \qquad (6)$$

Basically this latter relation comes from the fact that (2) represents a decomposition of the signal onto the improper basis of signals:

$$Z^{\xi}(f,\beta) = e^{-2i\pi\xi f} f^{-2i\pi\beta-r-1} \qquad (7)$$

which are eigenfunctions of (5).
These signals as well as the β variable get a geometrical interpretation when going to phase space, i.e. to the time-frequency half-plane ($f>0$). The problem of devising a suitable time-frequency representation of real signals has been solved elsewhere [3]. The relevant one in this context is given by:

$$P(t,f) = f \int_{\mathbb{R}} e^{-2i\pi uft} \left(\frac{u}{2sh(u/2)}\right)^{2r+2} S\left(\frac{fue^{-u/2}}{2sh(u/2)}\right) S^*\left(\frac{fue^{u/2}}{2sh(u/2)}\right) du \qquad (8)$$

Representation (8) possesses a unitarity property given by:

$$\int_{\mathbb{R}\times\mathbb{R}^+} P_1(t,f)\; P_2(t,f)\; f^{4r+2}\, dt\, df = |(S_1, S_2)|^2 \qquad (9)$$

where P_1 and P_2 correspond to the signals S_1 and S_2 respectively and where the r.h.s. is defined by (4).

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The correspondence $S \longrightarrow P$ has a remarkable covariance property with respect to the group $E = G \times \mathbb{R}$ where G is the affine group of elements (a,b). Indeed, let V(a,b,c) be the projective representation of $G \times \mathbb{R}$ defined by:

$$V(a,b,c)\, S(f) = a^{r+1}\, f^{2i\pi c}\, e^{-2i\pi abf}\, S(af) \qquad (10)$$

Then the following diagram is commutative:

$$\begin{array}{ccc} S(f) & \longrightarrow & V(a,b,c)S(f) \\ \downarrow & & \downarrow \\ P(t,f) & \longrightarrow & P(a^{-1}t-b-a^{-1}cf^{-1}, af) \end{array} \qquad (11)$$

Notice that E is the only 3-parameter group containing the affine group G as an invariant subgroup. This property together with (11) shows that V plays the same role here as the inhomogeneous metaplectic representation in the case of Wigner's function [4] . In the following E is called *compression group* to refer to its appearance in radar theory [5] .

Another property of P is obtained by inserting signals (7) into (8) which yields:

$$P(t,f) = f^{-2r-2}\, \delta(t - (\beta/f) - \xi) \qquad (12)$$

Thus the representation of signals Z^{ξ} is localized on hyperbolas in the time-frequency half-plane. This allows to interpret (7) as a realization of the so-called "Doppler-tolerant" signals [6].

In applications we are concerned with signals localized in a bounded domain of the time-frequency half-plane. For such signals the integral of P(t,f) on hyperbolas introduced in (12) must vanish for $|\beta|$ sufficiently large. According to (9) this implies that the Mellin transform of a signal localized in the time-frequency half-plane is itself localized. This remark is at the basis of the development of the discrete Mellin transform.

2. Some properties of the transform.

The correspondence between Hilbert spaces $L^2(\mathbb{R}_*^+, f^{r+1}df)$ and $L^2(\mathbb{R}, d\beta)$ given by formulas (1) and (2) possesses the same kind of properties as the Fourier transform. The analogy comes from the fact that the representation (10) considered for $b=\xi(1-a)$:

$$S(f) \longrightarrow S'(f) = a^{r+1}\, f^{2i\pi c}\, e^{-2i\pi\xi(1-a)f}\, S(af)$$

is an alternative representation of the Heisenberg group.

On f-space, an invariant product with respect to (5) can be introduced by:

$$(S_1 \circ S_2)(f) = f^{r+1}\, e^{2i\pi\xi f}\, S_1(f)\, S_2(f) \qquad (13)$$

The Mellin transform of this product is given by:

$$\mathcal{M}\,[S_1 \circ S_2] = \mathcal{M}\,[S_1] * \mathcal{M}\,[S_2] \qquad (14)$$

where the r.h.s. operation is the usual convolution of functions on β-space.

A multiplicative convolution of functions on $\mathbb{R}^+$ can also be introduced by:

$$(S_1 ** S_2)(f) = \int_{\mathbb{R}^+} S_1(f/f')\, S_2(f') \times e^{2i\pi\xi(f/f'+f'-f)} f'^{-1} df' \qquad (15)$$

For a given S_1 (or S_2), this is the most general linear operation commuting with transformation (5). The Mellin transform of (15) yields the relation:

$$\mathcal{M}\,[S_1 ** S_2] = \mathcal{M}\,[S_1]\; \mathcal{M}\,[S_2] \qquad (16)$$

where the r.h.s. operation is the usual product of functions in β-space.

For future discretization of the tranform, it is useful to define the geometric sampling distribution :

$$\Delta^{\xi,r}(f,A) = \sum_{n=-\infty}^{\infty} A^{-nr}\, e^{-2i\pi\xi A^n}\, \delta(f-A^n) \qquad (17)$$

whose Mellin transform is the distribution:

$$\mathcal{M}\,[\Delta] = (1/\ln A) \sum_{p=-\infty}^{\infty} \delta(\beta - p/\ln A) \qquad (18)$$

Expressions (17) and (18) are the counterparts of the "Dirac combs" correspondence in the Fourier transform. In the following, we will restrict ourselves to the case $\xi=0$ and drop altogether the indices ξ and r. The case $\xi \neq 0$ is not fundamentally different.

3. Discrete Mellin transform

We focalize now to signals localized in a bounded domain of the time-frequency half-plane (f>0). To any signal S(f) we associate the signal $\bar{S}(f)$ given by:

$$\bar{S}(f) = \sum_{n=-\infty}^{\infty} Q^{n(r+1)}\, S(Q^n f) \qquad (19)$$

In the following $\bar{S}(f)$ is called *dilatocycled form* of S(f). The use of the Δ distribution (17) permits a compact rewriting of (19) under the form:

$$\bar{S} = \Delta(f,Q) ** S(f) \qquad (20)$$

The Mellin transform of equation (20) is given by:

$$\underline{\mathcal{M}}(\beta) = \mathcal{M}\,[\Delta(f,Q)]\; \mathcal{M}(\beta) \qquad (21)$$

Recalling equation (18), we see that $\underline{\mathcal{M}}(\beta)$ is, up to a constant factor, the sampled form of the Mellin transform.

As noted at the end of Sec.1, every localized signal in the time-frequency plane is β-limited and this remark permits to periodize $\underline{\mathcal{M}}(\beta)$ by the operation:

$$\overline{\underline{\mathcal{M}}}(\beta) = \mathcal{M}\ [\Delta(f,q)] * \underline{\mathcal{M}}(\beta) \qquad (22)$$

In order to avoid aliasing, the parameter q must verify:

$$1/\ln q \geq |\beta_2 - \beta_1| \qquad (23)$$

where β_1 and β_2 are the extreme points of the support of $\underline{\mathcal{M}}$.

The f-form of (22) is obtained by inverting the Mellin transform using (2) and (16). The result is (cf.(13)):

$$\underline{\overline{S}}(f) = \Delta(f,q) \circ \overline{S}(f) \qquad (24)$$

If the real numbers Q and q appearing respectively in (20) and (22) are connected by the relation:

$$Q = q^N \qquad (25)$$

where N is a positive integer, then (22) is a sampled periodic function. In the same way condition (25) ensures that the geometric sampling (24) does not destroy the dilatocycled structure of the function $\overline{S}(f)$ defined in (19). The discrete Mellin transform is then readily obtained by writing explicitely (22) and (24). The result is:

$$\overline{\mathcal{M}}(p/\ln Q) = \ln Q\ (N)^{-1} \sum_{k=P}^{P+N-1} q^{k(r+1)}\ e^{2i\pi kp/N}\ \overline{S}(q^k) \qquad (26)$$

In(26), the integer P is determined by the support of S(f).

A synopsis of the operations leading to (26) is given in Table 1. The practical exploitation of the discretized formulas can be carried out with any FFT algorithm.

4. Applications in broad-band signal analysis.

The development of a fast mellin transform gives a tool for computation of signal functionals containing time dilations. This point is now briefly illustrated on the two problems relative to the implementation of the wide-band ambiguity function and of the affine time-frequency representation.

The general radar ambiguity function is defined by [7]:

$$|\chi(a,b)|^2 = a\left|\int_{\mathbb{R}} S(f)\ S^*(af)\ e^{2i\pi abf}\ df\right|^2$$

The computation of this expression is readily carried out using (6) and the unitarity relation (3). Results obtained with this procedure are shown in Fig.1.

A class of affine time-frequency distributions relevant to signal analysis [8] is given by:

$$P(t,f) = f\int_{\mathbb{R}} e^{2i\pi tf(\lambda(u)-\lambda(-u))} S(f\lambda(u)) S^*(f\lambda(-u)) \left|\frac{d}{du}\left(\lambda(u)-\lambda(-u)\right)\right| (\lambda(u)\lambda(-u))^{r+1} du \qquad (27)$$

Expression (8) corresponds to the special case:

$$\lambda(u) = u/(e^u - 1) \qquad (28)$$

To compute (27), it is convenient to perform first a Mellin transform with respect to the variable f. The result of the operation can be formulated in terms of the transforms of the two functions :

$$S_1(f,u) = (f\lambda(u))^{-r/2} e^{2i\pi tf\lambda(u)} S(f\lambda(u))$$

and

$$S_2(f,u) = (f\lambda(-u))^{-r/2} e^{-2i\pi tf\lambda(-u)} S(f\lambda(-u))$$

The computation is readily completed by using properties (6) and (14) and the final result in the time-frequency half-plane is obtained by inversion of a Mellin transform. Two applications of the procedure are given in Fig.2 for the choice (28) of the λ-function. The first corresponds to the hyperbolic signal (7) and the second to the "minimal" signal [9] S_{min} given by:

$$S_{min}(f) = f^{2\pi\gamma}\ e^{-2\pi\gamma f} \qquad (29)$$

where γ is a positive parameter adjusting the spectral width.

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$$S(f) = \int_{-\infty}^{\infty} \mathcal{M}(\beta)\, f^{-2i\pi\beta - r - 1}\, d\beta$$

$$\mathcal{M}(\beta) = \int_{0}^{\infty} S(f)\, f^{2i\pi\beta + r}\, df$$

Dilatocycling in f-space
⟷
Arithmetic sampling in β-space

Periodizing in β-space
⟷
Geometric sampling in f-space

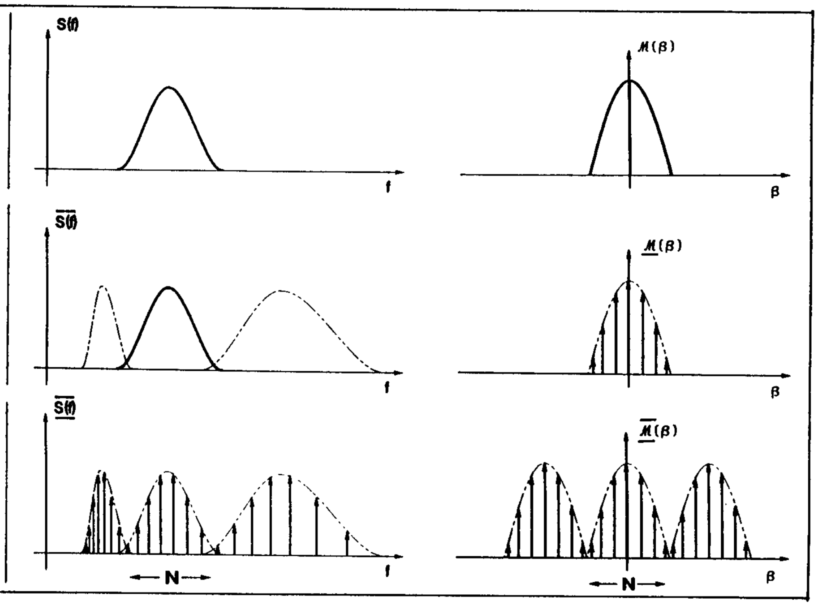

Table 1 : Graphical development of the discrete Mellin transform

(a) (b)

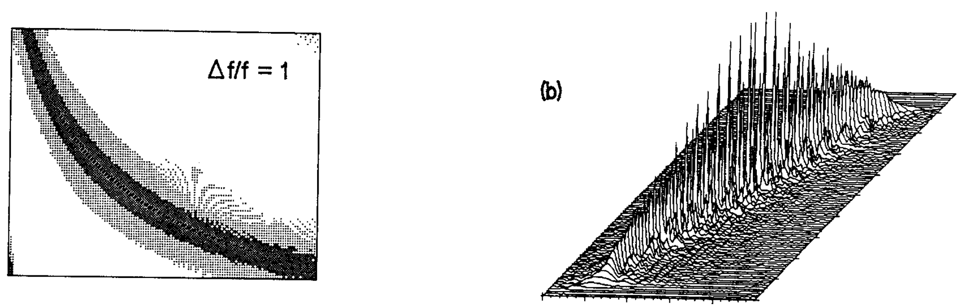

Fig 1 : Wide-band ambiguity functions :
a) Hyperbolic signal (7) , b) Radar code built with regular frequency steps

(a) (b)

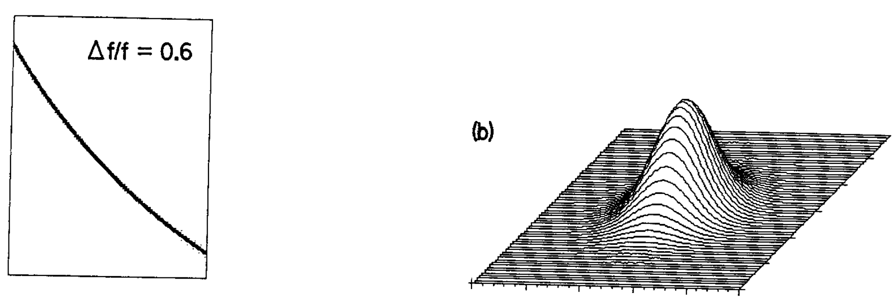

Fig 2 : Affine time-frequency representations (8) :
a) Hyperbolic signal (7) , b) Minimal signal (29)