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Improving portfolios global performance using a cleaned and robust covariance matrix estimate

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Abstract This paper presents how the use of a cleaned and robust covariance matrix estimate can improve significantly the overall performance of Maximum Variety and Minimum Variance portfolios. We assume that the asset returns are modelled through a multi-factor model where the error term is a multivariate and correlated elliptical symmetric noise extending the classical Gaussian assumptions. The factors are supposed to be unobservable and we focus on a recent method of model order selection, based on the Random Matrix Theory to identify the most informative subspace and then to obtain a cleaned (or de-noised) covariance matrix estimate to be used in the Maximum Variety and Minimum Variance portfolio allocation processes. We apply our methodology on real market data and show the improvements it brings if compared with other techniques especially for non-homogeneous asset returns.

Keywords Portfolio Selection · Maximum Variety Portfolio · Minimum Variance Portfolio · Covariance Matrix · Random Matrix Theory · Thresholding · Factor Model · Elliptic distribution

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1 Introduction

Modern portfolio theory introduced by Markowitz [32] lays the foundation for optimal portfolio construction with the so-called mean-variance strategy. This optimization problem maximizes the expected return for a given risk level in order to obtain the optimal weights. Nevertheless, its practical implementation relies on the knowledge of the empirical expected return, a quantity classically known to be very hard to estimate. To overcome these drawbacks, allocation methods focusing solely on the covariance matrix estimation have been developed, such as the Global Minimum Variance Portfolio (GMWP) or the Equally Risk Contribution Portfolio [10], [30].

An alternative method has been proposed in [8,9], based on portfolio diversification and having only the covariance matrix as an input parameter. This method seeks the most diversified portfolio by maximizing the variety (or diversification) ratio to reduce common risk exposures.

In financial modeling, a widely used estimator of the covariance matrix is the Sample Covariance Matrix (SCM), optimal under Gaussian assumptions. Nevertheless, it is well-known that asset returns usually exhibit departures from the optimal framework as asymmetry, fat tails, tail dependence, thus leading to large estimation errors. To deal with this point, covariance matrix estimation has been extended under non-Gaussian distributions [33, 48]. These robust estimators are generally adapted when $N > m$, where N is the sample size and m is the number of assets. Indeed, for singular covariance matrix estimate ($N < m$) regularization approaches are required and some authors have therefore proposed an hybrid robust shrinkage covariance matrix estimators [1, 7, 39] based on Tyler's robust M-estimator [48] and Ledoit-Wolf's shrinkage approach [29].

Recent works [7, 13, 39, 51] based on Random Matrix Theory (RMT) have therefore considered robust estimation when $N < m$. In [51], the covariance estimation approach is based on the Shrinkage-Tyler M-estimator and the authors show that applying an adapted estimation methodology leads to achieving superior performance over many other competing methods under the GMWP framework. Another way to lower the estimation errors of the covariance matrix is to distinguish the signal part from the noisy part using filters. It is now well documented in financial literature that the introduction of multiple sources of risks is a key factor to challenge the Capital Asset Pricing Model (CAPM) single market factor assumption [44]. Multi-factor models have therefore emerged based either on statistical factors or on observable factors [16, 17, 19, 42], and are designed to capture common risk factors (systematic risks). In this setup, the covariance matrix estimate of the assets depends solely of the systematic part of the risk, as in [16]. Statistical multi-factor models are also very interesting tools. Instead of choosing the factors among many others and from empirical studies, the factors are determined from the assets universe, using statistical methods. Whereas the principal component analysis may fail in distinguishing informative factors from the noisy ones, RMT helps identifying a solution to filter noise as in [25,26,40,41] by correcting the eigenvalues of the covariance matrix, thanks to the upper bound of the Marčenko-Pastur distribution [31]. This method called "Eigenvalue clipping" provides competitive out-of-sample results [3], even though in most cases only the first component (market factor) is detected which is not completely satisfactory. Other recent works [4, 5, 27] deal with the class of Rotational Invariant Estimators (RIE) that use all of the information on both eigenvectors and eigenvalues of the covariance matrix. The methodology proposed in [4] leads to portfolios having a lower volatility that those obtained when using SCM, Ledoit & Wolf (LW) and Eigenvalue clipping methods.

In this paper, we extend the results presented in [22] by considering that the assets returns might be non-homogeneously distributed. Indeed, as in [22], we assume that the asset returns are still modelled through a multi-factor model where the error term is a multivariate and correlated elliptical symmetric noise. Nevertheless, in our approach the whitening procedure is now applied by group of homogeneous assets and the final covariance estimate obtained only using the de-noised part of the observations as suggested in [45–47,49].

This article is organized as follows: section 2 introduces the selected methods of portfolio allocation for this paper: the Maximum Variety (or VarMax) portfolio and the Minimum Variance (or MinVar) portfolio. Section 3 presents the classical model and the related assumptions. Section 4 describes the covariance matrix estimation methodology. Section 5 provides empirical illustrations ascertaining the efficiency of the proposed method compared to the conventional ones. Section 6 concludes and discusses our results.

Notations We use bold and capital letters for matrices, and bold and lowercase letters for vectors. For any matrix $\mathbf{A}$, $\mathbf{A}^T$ is its transpose, $Tr(\mathbf{A})$ its trace and $\|\mathbf{A}\|$ is the spectral norm. For any vector $\mathbf{x}$ of size m, $\mathscr{L} : \mathbf{x} \mapsto \mathscr{L}(\mathbf{x})$ is defined as the associated symmetric square matrix of size m obtained through the Toeplitz operator: $([\mathscr{L}(\mathbf{x})]_{i,j}) = x_{|i-j|+1}$. For any square matrix $\mathbf{A} = [a_{i,j}]$ of size m, $\mathscr{T}(\mathbf{A})$ represents the matrix $\mathscr{L}(\check{\mathbf{a}})$ where $\check{\mathbf{a}}$ fulfills $\check{a}_i = (\sum_{j=i}^{m} a_{j,j-i+1})/m$. The notation "bp" stands for basis point and one basis point is equal to 0.01%.

2 Portfolio allocation

Portfolio allocation is a widely studied problem. Depending on the investment objective, the resulting portfolio allocation differs. In this section two allocation methods are described: the Maximum Variety process and the Global Minimum Variance one. Both of them depend on a single parameter that is the covariance matrix of the asset returns. In practice, the Minimum Variance portfolio is known to lead to low-diversified but performing portfolios over recent years reinforcing the low volatility anomaly concept, whereas the Maximum Variety process leads to well-diversified (by construction) but less performing portfolios.

2.1 Maximum Variety (or VarMax) Portfolio

We consider m financial assets used to build an investment portfolio perfectly characterized by the allocation vector $\mathbf{w} = [w_1, \dots, w_m]^T$ where w_i represents the proportion invested in asset i. In particular, we have $0 \le w_i \le 1 \ \forall i \in [1, m]$ and $\sum_{i=1}^{m} w_i = 1$. In [8], the authors provide a strong mathematical definition of portfolio diversification introducing the Variety Ratio ($\mathscr{VR}$) associated with $\mathbf{w}$ that is none other than the ratio of the weighted arithmetic mean of volatilities over the portfolio volatility:

$$\mathscr{VR}(\mathbf{w}, \Sigma) = \frac{\mathbf{w}^T \mathbf{s}}{(\mathbf{w}^T \Sigma \mathbf{w})^{1/2}}, \tag{1}$$

where Σ is the variance covariance matrix of the m assets and $\mathbf{s} = [\sqrt{\Sigma_{11}}, \dots, \sqrt{\Sigma_{mm}},]^T$ the m-vector of corresponding volatilities. Thus, the Maximum Variety (or VarMax) strategy, denoted by $\mathbf{w}^*_{vr}$, is obtained as the solution of the following optimization problem under convex constraints on weights

$$\mathbf{w}^*_{vr} = \underset{\mathbf{w}}{\operatorname{argmax}} \, \mathscr{VR}(\mathbf{w}, \Sigma). \tag{2}$$

The VarMax Portfolio verifies some interesting properties, as described in [9]:

- VarMax is invariant by duplication: if an asset is duplicated in the universe, then VarMax will be unchanged giving half the weight to each duplicated asset,
- VarMax stays unchanged if a positive linear combination of the assets of the universe is added as a new asset,
- any asset of the universe not held in VarMax is more correlated to the portfolio than to any asset of the portfolio. Furthermore, the more diversified a long-only portfolio is, the greater its correlation with VarMax.

VarMax portfolios are often considered as interesting diversifying investments with respect to the other investments. The above last property would therefore suggest that the other portfolios might then be weakly diversified portfolios.

2.2 Minimum Variance (or MinVar) Portfolio

The Global Minimum Variance Portfolio (or GMVP) is obtained by computing the portfolio whose m-vector of weights $\mathbf{w}_{gmvp}$ minimizes the variance of the final portfolio. It can be formulated as a quadratic optimization problem including the linear constraint that the sum of the weights is equal to 1:

$$\min_{\mathbf{w}} \sigma^2(\mathbf{w}, \Sigma) = \min_{\mathbf{w}} \mathbf{w}^T \Sigma \mathbf{w}, \quad \text{s.t. } \mathbf{w}^T \mathbf{1}_m = 1 \tag{3}$$

with $\mathbf{1}_m$ being a m-vector of ones. The solution to (3), when there is no other constraint on the weight values, is then: $\mathbf{w}_{gmvp} = \dfrac{\Sigma^{-1}\mathbf{1}_m}{\mathbf{1}_m^T \Sigma^{-1} \mathbf{1}_m}$, and the corresponding portfolio variance writes $\sigma^2(\mathbf{w}_{gmvp}, \Sigma) = \dfrac{1}{\mathbf{1}_m^T \Sigma^{-1} \mathbf{1}_m}$.
As for the VarMax portfolio, the covariance matrix needs to be estimated. If we denote $\widehat{\Sigma}$ an estimate of Σ, then we have:

$$\widehat{\mathbf{w}}_{gmvp} = \frac{\widehat{\Sigma}^{-1}\mathbf{1}_m}{\mathbf{1}_m^T \widehat{\Sigma}^{-1} \mathbf{1}_m}.$$

In [52], the authors derive an optimal optimization strategy in order to minimize the realized portfolio variance, under an assumption of spiked structures[1] of both Σ and Σ^{-1}. In our case, the weights have to be positive, so that the optimal minimum variance portfolio weights cannot be obtained in a closed form expression. We will nevertheless compare several competing methods of covariance matrix estimation in order to get the GMVP.

To get solutions for (2) and (3), the unknown covariance matrix Σ has to be determined or estimated. This is a challenging problem in portfolio allocation due to the strong sensitivity of the optimisation process to outliers and estimation errors. Apart from the classical SCM or the Minimum Covariance Determinant (MCD, [43]) that is a method robust to outliers, reside subspace methods that aim at separating the signal space from the noise space, using the eigendecomposition of the SCM. The noise and signal subspaces are usually identified according to the eigenvalues magnitudes: the eigenvectors related to the lowest ones represent the noise whereas those related to the highest eigenvalues identify the signal. But the open question remains how to choose the separating threshold? In this paper we propose a robust and original technique that applies the Random Matrix Theory (RMT) results on the eigen-decomposition of a robust M-estimator leading to a denoised and robust covariance matrix estimate.

3 Model and assumptions

Let us assume that the investment universe contains m assets whose returns at each time $t = 1, \cdots, N$ are stored in the m-vector $\mathbf{r}_t$. We suppose also that $\mathbf{r}_t$ admits a K factors structure, where the $K < m$ common factors are unknown, and that the additive noise is a multivariate Elliptical Symmetric noise [23, 37] . The assumed model for $\mathbf{r}_t$ writes as follows:

$$\mathbf{r}_t = \mathbf{B}_t \mathbf{f}_t + \sqrt{\tau_t}\, \mathbf{C}^{1/2} \mathbf{x}_t, \tag{4}$$

where

- $\mathbf{r}_t$ is the m-vector of returns at time t,
- $\mathbf{B}_t$ is the $m \times K$-matrix of coefficients that define the assets sensitivities to each factor at time t,
- $\mathbf{f}_t$ is the K-vector of random factor values at t, supposed to be common to all the assets,
- $\mathbf{x}_t$ is a m-vector of independent Gaussian white noise with unit variance and non-correlated with the factors, i.e. $\mathbb{E}[\mathbf{x}_t \mathbf{f}_t^T] = \mathbf{0}_{m \times K}$,
- $\mathbf{C}$ is called the $m \times m$ scatter matrix that is supposed to be Toeplitz[2] structured [18] and time invariant over the period of observation,
- τ_t is a family of i.i.d positive random variables with expectation τ that is independent of the noise and the factors and drives the variance of the noise. These random variables are time-dependent and generate the Elliptical distribution [6] of the noise.

The Toeplitz assumption made on $\mathbf{C}$ is a required assumption for the proposed methodology described in section 4.1. This hypothesis imposes a particular structure for the covariance matrix of the additive noise, and is generally used to describe stationary processes [18]. In the case of model (4) this hypothesis is plausible as it states that the

[1] A spiked structure denotes a covariance model where some eigenvalues are located out of the "bulk", like outliers.

[2] A Toeplitz matrix is a diagonal-constant matrix.

additional white noise admits a Toeplitz-structured covariance matrix. In the case of financial time series where we only observe one sample at each time, the stationarity of the dependence structure of the assets is a statistical hypothesis really difficult to test in practice. This motivates the extension we propose in this paper, described in section 4.4, to splitting the assets universe into groups composed of assets having similar distributions, and being most probingly sampled from a stationary process representing a unique distribution for each group.

Given equation (4) the covariance matrix writes for a fixed period of time t:

$$\Sigma_t = \mathbf{B}_t \, \Sigma_t^f \, \mathbf{B}_t^T + \tau \, \mathbf{C}, \tag{5}$$

that is a $m \times m$-matrix composed of two terms: the factor-related term with $\Sigma_t^f = \mathbb{E}[\mathbf{f}_t \, \mathbf{f}_t^T]$ being of rank K, and the noise-related term being of rank m. Subspace methods aim at identifying the K highest eigenvalues of Σ_t supposed to represent the K- factors especially when the power of the factors is higher than the noise power.

Determining K, the number of factors is a tough task in all the model order selection problems, like e.g. when estimating the number of emitting sources in any received signal or when trying to unmix sources in hyperspectral images [2]. In financial applications, the K factors serve in building portfolios and also to identify the main sources of risks within the investment universe under study [14, 15, 21, 35], and is therefore of main importance in such cases.

In the next section we give a detailed description of our methodology that combines the robust Tyler M-estimator of the covariance matrix and the RMT results adapted to the above non Gaussian and multivariate model.

4 Proposed Methodology

4.1 General framework

The Tyler M-estimator [48] of the covariance matrix for the m-vector $\mathbf{r}_t$ is defined as being the solution of the following fixed-point equation:

$$\mathbf{X} = \frac{m}{N} \sum_{t=1}^{N} \frac{\mathbf{r}_t \, \mathbf{r}_t^T}{\mathbf{r}_t^T \, \mathbf{X}^{-1} \, \mathbf{r}_t}, \tag{6}$$

where the trace of the resulting matrix is equal to m, and N is the number of observations for $\mathbf{r}_t$. Applied to model (4) and under non-Gaussian assumptions, the resulting Tyler-M estimate (that we denote $\widehat{\mathbf{C}}_{tyl}$) is shown to be the most robust covariance matrix estimator [38, 48] for the true scatter matrix $\mathbf{C}$. $\widehat{\mathbf{C}}_{tyl}$ is also independent of the distribution of the variable τ.

Under the white noise assumption, extracting information from the observed signal using RMT is quite straightforward and has been proposed in many applications, like in source detection [24], in radar detection [12], or signal subspace estimation using an adapted MUSIC (MUltiple SIgnal Classification) detection algorithm [20]. Nevertheless, when the noise is correlated, RMT results do not apply directly as the variance of the Marčenko-Pastur threshold has to be estimated, and only numerical methods can help in finding the resulting threshold [11, 49]. In some cases, secondary data that do not contain any sources can be used to estimate the covariance matrix and then whiten the observed data.
However, recent works [45–47] brought a solution that consists of applying a biased Toeplitz operator on $\widehat{\mathbf{C}}_{tyl}$, let us say $\widetilde{\mathbf{C}}_{tyl} = \mathscr{T}\left(\widehat{\mathbf{C}}_{tyl}\right)$, which was proven to spectrally converge towards the theoretical scatter matrix $\mathbf{C}$. This result refers to the *Consistency Theorem* in [45–47], and asserts that whenever the sources are present in the observations, the resulting scatter matrix estimate is a consistent estimation of its theoretical value.
The first step of our methodology consists therefore in estimating $\widetilde{\mathbf{C}}_{tyl}$ from N observations of $\mathbf{r}_t$ in order to whiten the observations leading to the N whitened observations $\mathbf{r}_{w,t} = \widetilde{\mathbf{C}}_{tyl}^{-1/2} \, \mathbf{r}_t$.

Given the whitened observations $\{\mathbf{r}_{w,t}\}$ and their Tyler's covariance matrix $\widehat{\Sigma}_w$, it has been shown in [47] that the eigenvalues distribution of $\widehat{\Sigma}_w$ fit the predicted bounded distribution of Marčenko-Pastur [31]. However, if one or several sources are contained in the observations, being powerful enough to be detected, then there will be as many eigenvalues as there are sources standing outside the upper bound of the Marčenko-Pastur distribution, given in that case by $\bar{\lambda} = \sigma^2 \, (1 + \sqrt{c})^2$ where $c = m/N$ and $\sigma^2 = 1$ (due to the preceding whitening process σ^2 is equal to one). Once the K largest eigenvalues larger than $\bar{\lambda}$ are detected, we process as for the Eigenvalue clipping in [26] to set the values of the remaining $m-K$ lowest eigenvalues to a unique value equal to $\left(Tr\left(\widehat{\Sigma}_w\right) - \sum_{k=K+1}^{m} \lambda_k\right)/(m-K)$. Using also the corresponding eigenvectors, we then build back the de-noised assets covariance matrix to be used in (2) and (3) or in any other objective function. The whitening procedure is detailed more precisely in the next subsection.

4.2 Detailed whitening procedure

Given $\mathbf{R}$ the $m \times N$ matrix of observations, the de-noised covariance matrix estimate $\widehat{\Sigma}_w$ is obtained through the following procedure steps:

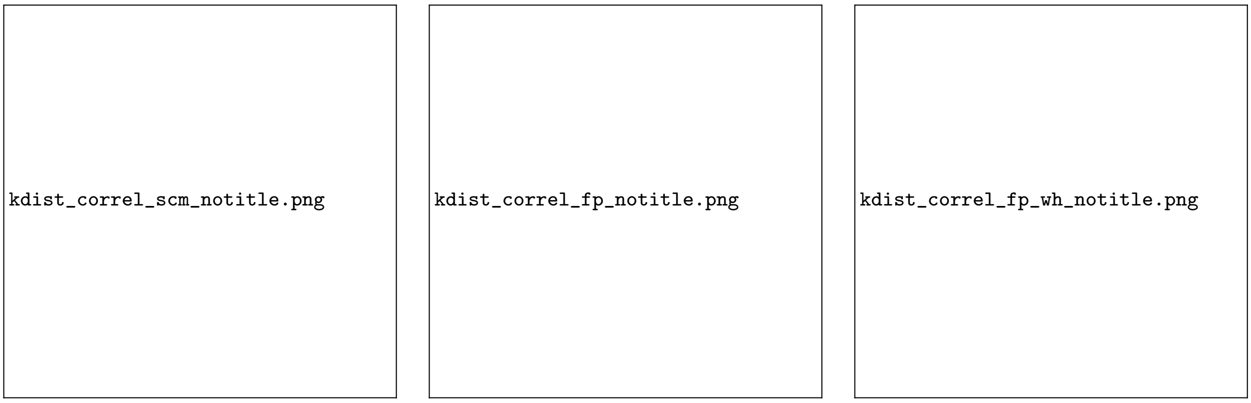

Fig. 1 Distributions of the logarithm of the eigenvalues of three covariance matrix estimates. Left side: Eigenvalues (log) of the SCM of the observations; Middle: Eigenvalues (log) of the Tyler covariance matrix of the observations; Right side: Eigenvalues (log) of the Tyler covariance matrix of the whitened observations. Observations contain $K = 3$ sources embedded in a multivariate K-distributed noise with shape parameter $\nu = 0.5$, and a Toeplitz coefficient $\rho = 0.8$. $m = 100, N = 1000$ ($c = 0.1$), and the (log) Marčenko-Pastur upper bound is here: $\log(\bar{\lambda}) = \log(1.7325)$.

S1 Set $\widehat{\mathbf{C}}_{tyl}$ as the Tyler-M estimate of $\mathbf{R}$, solution of (6),

S2 Set $\widetilde{\mathbf{C}}_{tyl} = \mathscr{T}\left(\widehat{\mathbf{C}}_{tyl}\right)$, the Toeplitz rectification matrix built from $\widehat{\mathbf{C}}_{tyl}$ for the Toeplitz operator $\mathscr{T}()$,

S3 Set $\mathbf{R}_w = \widetilde{\mathbf{C}}_{tyl}^{-1/2}\mathbf{R}$, the $m \times N$ matrix of the whitened observations,

S4 Set $\widehat{\Sigma}_{tyl}$ as the Tyler-M estimate of $\mathbf{R}_w$, solution of (6),

S5 Set $\widehat{\Sigma}_{tyl}^{clip} = \mathbf{U}\Lambda^{clip}\mathbf{U}^T$ where $\mathbf{U}$ is the $m \times m$ eigenvectors matrix of $\widehat{\Sigma}_{tyl}$ and Λ^{clip} is the $m \times m$ diagonal matrix of the eigenvalues $(\lambda_k^{clip})_{k\in[1,m]}$ corrected using the Eigenvalue clipping method [26] described in details in Appendix A.1,

S6 Finally, $\widehat{\Sigma}_w = \left(\widetilde{\mathbf{C}}_{tyl}^{1/2}\right)\widehat{\Sigma}_{tyl}^{clip}\left(\widetilde{\mathbf{C}}_{tyl}^{1/2}\right)^T$.

4.3 Simulation example

To illustrate the efficiency of the whitening process, we ran the following test: we simulate $N = 1000$ observations of a $m = 100$ here, sampled from a highly correlated K-distributed process [37] having a shape parameter $\nu = 0.5$, and a Toeplitz-structured covariance matrix whose coefficient $\rho = 0.8$ (each element i, j of the Toeplitz matrix is defined by $\rho^{|i-j|}, i, j = 1, \cdots, m$). We then embed $K = 3$ sources of information in the non-Gaussian and correlated noise, and we compare the eigenvalues distribution of the observations with the Marčenko-Pastur upper bound when the eigenvalues are computed from i) the SCM (on the left), ii) the Tyler M-estimate matrix (in the middle), and iii) the Tyler M-estimate matrix of the whitened observations. It appears clearly that the $K = 3$ factors can be identified quite easily only in the case where the observations are firstly whitened.

4.4 The case of non-homogeneous assets returns

The whitening process proposed above is made under the implicit assumption that the assets returns are drawn from a unique multivariate law and are therefore homogeneous in law. As described hereafter this assumption is unrealistic for financial time series of returns. We therefore propose to split the m assets into $p < m$ groups, each composed of $\{m_q\}_{q=1}^p$ assets (with $\sum_{q=1}^p m_q = m$), and formed to be composed of assets having similar distributions. We set a fixed number of groups, and group the assets regarding their returns distributions. Under this new assumption, model (4) applies for each group q as follows:

$$\mathbf{r}_t^{(q)} = \mathbf{B}_t^{(q)}\mathbf{f}_t + \sqrt{\tau_t}\,\mathbf{C}_{(q)}^{1/2}\mathbf{x}_t, \tag{7}$$

Then, the full model (4) rewrites:

$$\begin{bmatrix} \mathbf{r}_t^{(1)} \\ \vdots \\ \mathbf{r}_t^{(p)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_t^{(1)} \\ \vdots \\ \mathbf{B}_t^{(p)} \end{bmatrix} \mathbf{f}_t + \sqrt{\tau_t} \begin{pmatrix} \mathbf{C}_{(1)} & \mathbf{0}_{1,2} & \cdots & \mathbf{0}_{1,p} \\ \mathbf{0}_{2,1} & \mathbf{C}_{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0}_{p-1,p} \\ \mathbf{0}_{p,1} & \cdots & \mathbf{0}_{p,p-1} & \mathbf{C}_{(p)} \end{pmatrix}^{1/2} \mathbf{x}_t, \tag{8}$$

where $\mathbf{0}_{i,j}$ denotes the null matrix of size $m_i \times m_j$, $i, j = 1, \cdots, p$, corresponding to the additional hypothesis that the groups are uncorrelated each others. The complete scatter matrix $\mathbf{C}$ is therefore block-constructed, and block-Toeplitz.

To form the groups of assets at each date t and given a past period of N observations, we proceed as follows:

- for each asset i, we compute the sample mean μ_i and the sample standard deviation σ_i using its N returns $\mathbf{r}_i$,
- we compute the "standardized" returns $\tilde{\mathbf{r}}_i = (\mathbf{r}_i - \mu_i)/\sigma_i$,

- we compute several quantiles from $\tilde{\mathbf{r}}_i$, and append μ_i and σ_i to the vector of the computed quantiles to get our variables on which to group the assets, and finally,
- we use the classical Ascending Hierarchical Classification (AHC) using the Euclidean distance and the Ward measure [50] to form the p groups.

AHC is a very classical classification method but ensures to get homogeneous groups for which the intra-group variances are smaller than the inter-group variances.

The three first steps **S1**, **S2** and **S3** of the whitening process described in 4.2 are therefore repeated for each group (q), $q = 1, \cdots, p$: given $\mathbf{R}^{(q)}$ the $m_q \times N$ matrix of observations for assets in group (q), going through **S1** to **S3** leads to $\mathbf{R}_w^{(q)}$ the whitened matrix of observations for group (q). Once $\mathbf{R}_w$ has been completed, then steps **S4** to **S6** are applied and lead to the block-constructed covariance matrix estimate $\widehat{\Sigma}_w$. This is a mixed version between a global whitening process and a diagonal whitening process (applied when the series are only standardized). Our process can be viewed as a block-diagonal whitening process and ensures that the whitened groups are more homogeneous than the overall group of assets.

5 Application

In this section we apply our methodology to the Maximum Variety and Minimum Variance portfolios. The allocation is done over a blend of European equities[3]. This universe is composed of twenty-four sectors, thirteen countries and six smart beta indexes. Using a blend of equities instead of individual stock allows capturing collective risks (systematic) rather than idiosyncratic ones and reinforce portfolio diversification without having to impose constraints to reduce a stock-specific and liquidity risk. Our daily track record spans from July 2000, the 27th to May 2019, the 20th. We use closing prices, i.e. the last traded price during stock exchange trading hours.
To build the portfolios, the weights are computed as follows: we estimate every four weeks the covariance matrix of assets using the last year of daily returns (N = 260 weekdays) and we optimize the objective function of Maximum Variety (1) or Minimum Variance (3) to obtain the vector of weights. Finally, the weights remain constant between two rebalancing periods of four weeks. We apply our methodology in two manners: the first one, named "RMT-Tyler-Wh", contains the whitening process applied on the universe as a whole, whereas the one denoted by "RMT-Tyler-Wh-by-Gr" refers to the whitening process applied on each group of assets[4].

We compare the results with those obtained using the "SCM" and also with three other competing methods: the first one, denoted as "RMT-SCM" uses the Eigenvalue clipping of [26], the second one, that we denote as "LW", is the method that uses the Ledoit & Wolf shrinkage of [28], and finally the method using the Rotational Invariant Estimator of [4, 5], denoted as "RIE". These methods are briefly described in appendix.

We report several portfolios statistics computed over the whole period in order to quantify the benefits of the proposed methodology: the annualized return, the annualized volatility, the ratio between the annualized return and the annualized volatility, the value of the maximum drawdown (that is the return between the highest and the lowest portfolio levels observed during the whole period), and the average of the Variety Ratios computed at each rebalancing date. The higher is the return/volatility ratio, the lower is the maximum drawdown and the higher is the variety ratio, and better performing is the portfolio. Performances are also compared to the performance of the MSCI® Europe Index [36] (composed of large and mid cap equity stocks across 15 countries of the European regions), and to the performance of the equi-weighted portfolio, composed of all the assets that are equally weighted.

5.1 Variety Maximum (or VarMax) portfolios results

Figure 2 shows the evolution of the VarMax portfolios wealth, starting at 100 at the beginning of the first period.
The "SCM", "RMT-SCM", "LW", "RIE", "RMT-Tyler-Wh" and "RMT-Tyler-Wh-by-Gr" VarMax portfolios are respectively in red, dashed red, dash-dotted blue, blue, purple, and green lines. The naive equi-weighted portfolio is reported as the dotted black line, and the price of the benchmark, also rebased at 100 at the beginning of the period, is the black line.

The proposed "RMT-Tyler-Wh"-based techniques clearly outperform the conventional ones. Moreover, whitening homogeneous groups of data instead of the whole data set improves even more the results. Regarding the other methods, "RMT-SCM" is the only one that outperforms significantly "SCM", but shows weaker performances than our proposed method does; "LW" and "RIE" are quite similar to "SCM".

On the figure we have reported the "net of transaction fees" portfolios wealth, considering 0.07% of fees (or 7 ba-

[3] Data are available upon request.

[4] The number of group is $p = 6$ and the quantiles used are q_θ and $q_{1-\theta}$ with $\theta \in [1\%, 2.5\%, 5\%, 10\%, 15\%, 25\%, 50\%]$.

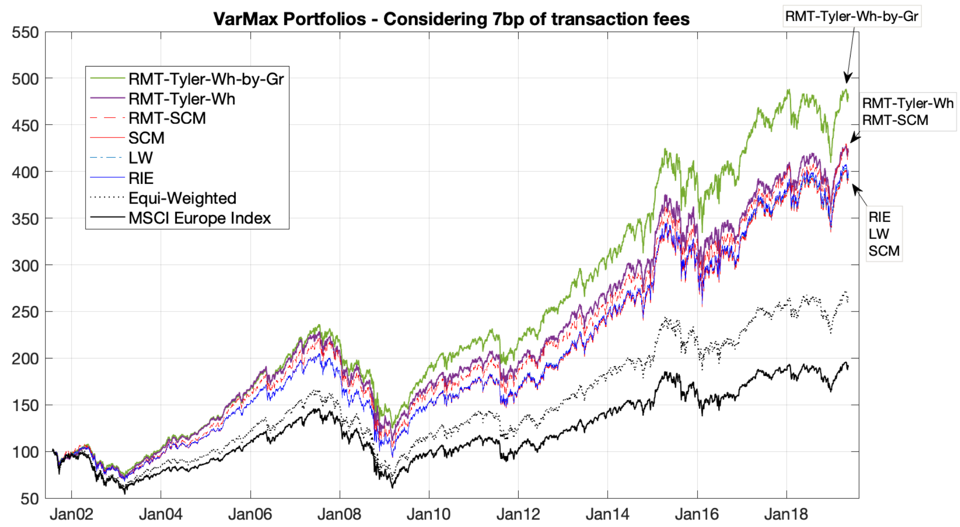

Fig. 2 VarMax portfolios wealth from July 2001 to May 2019. The proposed "RMT-Tyler-Wh-by-Gr" (green line) leads to improved performances vs the "RMT-Tyler-Wh" (purple), the "RMT-SCM" (dashed red), the "LW" (dash-dotted blue), the "RIE" (blue) and the "SCM" (red), as shown in Table 1: higher annualized return, lower annualized volatility, lower maximum drawdown and higher Diversification Ratio. But it results in a twice higher turnover: we then have taken into account 7bp (or 0.07%) of transactions fees to compare the portfolios wealth.

sis points denoted as "bp") applied to any weight change from one time to the next one. Measuring the total weights changes is referred as the turnover of the portfolio. We assume that the turnover between two consecutive periods t and $t+1$ is measured by $\sum_{i=1}^{m} |w_{i,t+1} - w_{i,t}|$. If, for example, the turnover is equal to 0.15 for changing weights from t to $t+1$, then the portfolio performance computed between t and $t+1$ will be decreased by 0.15×7 bp $= 0.0105\%$. Turnover is an important number in portfolio allocation. If you ever find an apparently well performing strategy that indicates you to change the overall portfolio at each time, then the cost of changing the overall portfolio will surely be equivalent or larger than would be the performance of the strategy itself. Here, the proposed technique leads to increase the cumulated turnover, but reasonably enough to let the improvement be a significant improvement that do not cost all the benefits of the technique. Limiting the turnover is often added as an additional non linear constraint to any optimization process like (2) or (3).

We finally report on Table 1 some statistics on the overall portfolios performance: we compare, for the whole period, the annualized return, the annualized volatility, the ratio between the return and the volatility, the maximum drawdown and the average value of the diversification ratio, for the portfolios and the benchmark. All the indicators related to the proposed technique show a significant improvement with respect to the other methods: a higher annualized return, a lower volatility (so a higher return/volatility ratio), a lower maximum drawdown and a higher diversification ratio.

5.2 Minimum Variance (or MinVar) portfolios results

Results obtained for the MinVar portfolios also show some improvements but less important than for the VarMax portfolios. Figure 3 shows that whitening by groups ("RMT-Tyler-Wh-by-Gr") improves the performance whereas whitening the whole assets ("RMT-Tyler-Wh") do not bring improvement with respect to all the other approaches, even if the variety ratio is higher. "RMT-SCM", "LW" and "RIE" provide lower or similar performances if compared to "SCM". Minimizing the portfolio variance leads to choosing the assets having the lowest volatilities. Then, using a robust approach does flatten the volatility differences between assets and then the ex-post portfolio volatility, computed classically, will be higher than the ex-post portfolio volatility computed using the robust matrix. Nevertheless, our process leads to higher performance that the classical SCM exhibiting a higher diversification ratio, and also a lower maximum drawdown.

To illustrate this purpose, Figure 4 plots the standard deviations of the invested assets versus the resulting weights obtained for MinVar/SCM weights (on the top graph) the VarMax/SCM (on the bottom graph). The same conclusion arises for the "RMT-Tyler-Wh-by-Gr". It shows explicitly which assets are preferred and when, according to their volatility level. On a similar way, Figure 5 shows that VarMax assigns non-zeros weights to the less correlated assets if compared to the non-zeros MinVar weights.

As for the VarMax portfolios, Table 2 reports the MinVar portfolios statistics. Again, the indicators related to the

VarMax Portfolios	Annualized Return	Annualized Volatility	Ratio (Return / Volatility)	Maximum Drawdown	Diversification Ratio (avg)
RMT-Tyler-Wh-by-Gr	9.65%	12.03%	0.80	46.84%	1.57
RMT-Tyler-Wh	8.90%	13.16%	0.68	51.18%	1.44
RMT-SCM	8.94%	13.79%	0.65	54.15%	1.27
RIE	8.65%	13.65%	0.63	54.44%	1.38
LW	8.59%	13.57%	0.63	54.28%	1.40
SCM	8.56%	13.68%	0.63	54.45%	1.38
Equi-Weighted	*6.60%*	*15.37%*	*0.43*	*57.82%*	*1.19*
Benchmark	*4.71%*	*14.87%*	*0.32*	*58.54%*	

Table 1 Some performance numbers for VarMax portfolios with 0.07% of fees from July 2001 to May 2019. The results are ranked in descending order according to the ratio (Return / Volatility).

proposed technique show an improvement if compared to the classical techniques.

6 Conclusion

In this paper, we have shown that when the covariance matrix is estimated with the Tyler M-estimator and RMT, the Maximum Variety and the Minimum Variance Portfolio allocation processes lead to improved performances with respect to several classical estimators. Moreover, we have proposed to extend the first results in [22] by considering the case of non-homogeneous asset returns while keeping a multifactor model where the error term is a multivariate and correlated elliptical symmetric noise. Indeed, the underlying assumption of the whitening process is that asset returns are homogeneous in distribution, which is unrealistic for financial time series of returns. To deal with this point, we have first grouped the assets within homogeneously distributed classes before processing. Applying the whitening process on homogeneous groups of data rather than the whole data set improves even more the results. This paper has focused on both the Maximum Variety and Minimum Variance portfolios but can be applied on other allocation framework involving covariance matrix estimation (and/or model order selection). Finally, the main factors identified by the whitening process can also be used and offer many possible avenues for future research, such as creating dynamic factor portfolios or reducing the dimension of the covariance matrix when $N < m$.

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Conflict of interest

The authors declare that they have no conflict of interest.

Appendix: Brief description of alternative covariance matrix estimators

Here, we briefly introduce some well-known covariance matrix estimators. In the following, $c = m/N$ and $\widehat{\mathbf{E}} = \widetilde{\mathbf{R}}\widetilde{\mathbf{R}}^T/N$ is the standardized SCM where $\widetilde{\mathbf{R}} = (\widetilde{\mathbf{r}}_i)_{i\in[1,m]}$ as defined in section 4.4.

A.1 Eigenvalue clipping (or RMT-SCM)

Laloux et al. [26] proposed Eigenvalue clipping in order to separate signal and noise subspaces using Marčenko-Pastur [34] boundary properties of the eigenvalues. The Eigenvalue clipping estimator of $\widehat{\mathbf{E}}$ is as follows:

$$\widehat{\mathbf{E}}_{clip} = \sum_{k=1}^{m} \lambda_k^{clip} \mathbf{u}_k \mathbf{u}_k^T$$

with $\mathbf{u}_k$ the eigenvector associated to the eigenvalue λ_k of $\widehat{\mathbf{E}}$, and λ_k^{clip} defined as follows:

$$\lambda_k^{clip} = \begin{cases} \lambda_k, \text{ if } \lambda_k \geq (1+\sqrt{c})^2 \\ \widetilde{\lambda}, \text{ otherwise} \end{cases} \tag{9}$$

where $\widetilde{\lambda}$ is chosen such that $Tr(\widehat{\mathbf{E}}_{clip}) = Tr(\widehat{\mathbf{E}})$.

A.2 Ledoit & Wolf shrinkage (or LW)

Ledoit & Wolf [28] introduced some shrinkage estimators particularly adapted to financial asset returns and based on the single factor model of Sharpe [44], where the factor is a market index. LW is a linear combination of the SCM and the covariance matrix containing the market information. This model can be written as follows:

$$r_{i,t} = \alpha_i + \beta_i F_t + \varepsilon_{i,t}, \forall i \in [1,m] \text{ and } \forall t \in [1,N] \tag{10}$$

where $r_{i,t}$ is the return of stock i at time t , α_i is the active return of the asset i, F_t is the market index return at time t, β_i is the asset sensitivity to the market index return, and $\varepsilon_{i,t}$ is the idiosyncratic return for asset i at t. This latter term is assumed to be uncorrelated to the market index. Then the covariance matrix writes:

$$\mathbf{M}_r = \sigma_F^2 \beta \beta^T + \Omega_\varepsilon$$

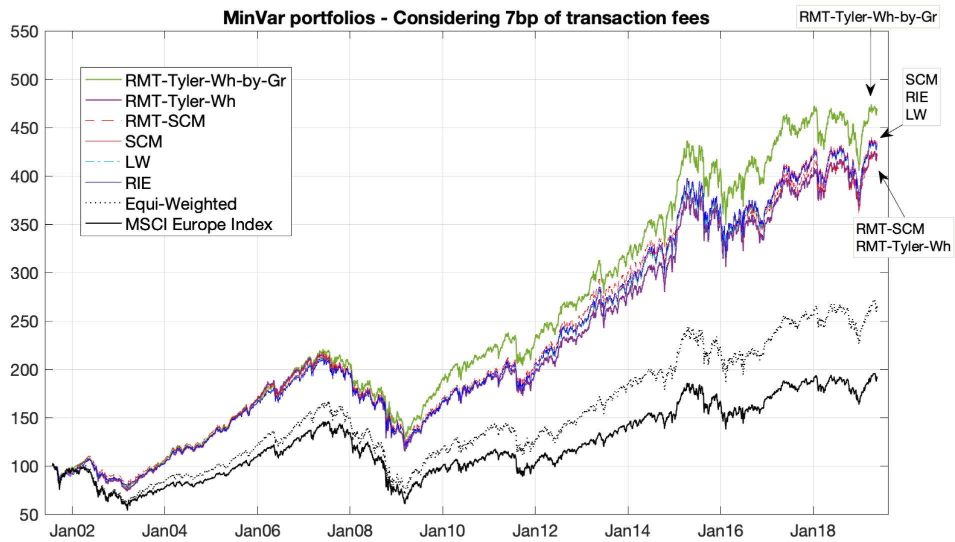

Fig. 3 MinVar portfolios wealth from July 2001 to May 2019. The proposed "RMT-Tyler-Wh-by-Gr" (green line) leads to improved performances vs the "RMT-Tyler-Wh" (purple), the "RMT-SCM" (dashed red), the "LW" (dash-dotted blue), the "RIE" (blue) and the "SCM" (red), as shown in Table 2. MinVar portfolios are known to result in poorly diversified portfolios and to invest in the lowest volatile assets. But surprisingly, the low-volatility anomaly applies in such cases.

MinVar Portfolios	Annualized Return	Annualized Volatility	Ratio (Return / Volatility)	Maximum Drawdown	Diversification Ratio (avg)
RMT-Tyler-Wh-by-Gr	9.35%	11.08%	0.84	41.07%	1.52
LW	8.75%	10.75%	0.81	43.69%	1.21
RIE	8.76%	10.78%	0.81	43.24%	1.19
SCM	8.74%	10.92%	0.80	43.78%	1.19
RMT-SCM	8.62%	10.80%	0.80	43.95%	1.14
RMT-Tyler-Wh	8.72%	11.58%	0.75	46.50%	1.36
Equi-Weighted	*6.60%*	*15.37%*	*0.43*	*57.82%*	*1.19*
Benchmark	*4.71%*	*14.87%*	*0.32*	*58.54%*	

Table 2 Some performance numbers for MinVar portfolios with 0.07% of fees from July 2001 to May 2019. The results are ranked in descending order according to the ratio (Return / Volatility).

with $\beta = [\beta_1, \cdots, \beta_m]^T$, σ_F^2 is the variance of the market returns and Ω_ε the covariance matrix of the idiosyncratic error.
An estimator for $\mathbf{M}_r$ can be determined:

$$\widehat{\mathbf{M}}_r = \hat{\sigma}_F^2 \,\widehat{\beta}\,\widehat{\beta}^T + \widehat{\Omega}_\varepsilon$$

where each $\hat{\beta}_i$ is estimated individually using the OLS estimator based on equation (10) and the $\widehat{\Omega}_\varepsilon$ is a diagonal matrix composed of the OLS residual variances. Finally, $\hat{\sigma}_F^2$ is the sample variance of the market returns.

The Shrinkage-to-Market estimator from Ledoit & Wolf is therefore equal to:

$$\widehat{\Sigma}(\gamma) = \gamma \widehat{\mathbf{M}}_r + (1-\gamma)\,\mathbf{S}$$

where $\gamma \in [0,1]$ is the shrinkage parameter estimated as in [28], and $\mathbf{S}$ is the SCM of asset returns.

A.3 Rotational invariant estimator (or RIE)

Bun et al. [4, 5] proposed an optimal rotational invariant estimator for general covariance matrices by computing the overlap between the true and sample eigenvectors introduced first by Ledoit & Péché [27]. For large m, the optimal rotational invariant estimator (RIE) of $\widehat{\mathbf{E}}$ is as follows:

$$\widehat{\mathbf{E}}_{RIE} = \sum_{k=1}^{m} \lambda_k^{RIE}\, \mathbf{u}_k \mathbf{u}_k^T$$

with $\mathbf{u}_k$ the eigenvector associated to the eigenvalue λ_k of $\widehat{\mathbf{E}}$, and λ_k^{RIE} defined as follows:

$$\lambda_k^{RIE} = \frac{\lambda_k}{|1 - c + c\, z_k\, s(z_k)|^2}$$

where $z_k = \lambda_k - iN^{-1/2}$ is a complex number and $s(z)$ denotes the discrete form of the limiting Stieltjes transform

$$s(z) = \frac{1}{m} \sum_{j=1}^{m} \frac{1}{z - \lambda_j}$$

We also ensure that $Tr(\widehat{\mathbf{E}}_{RIE}) = Tr(\widehat{\mathbf{E}})$. For this purpose, we multiply each λ_k by ν with $\nu = \sum_{k=1}^{m} \lambda_k / \sum_{k=1}^{m} \lambda_k^{RIE}$.

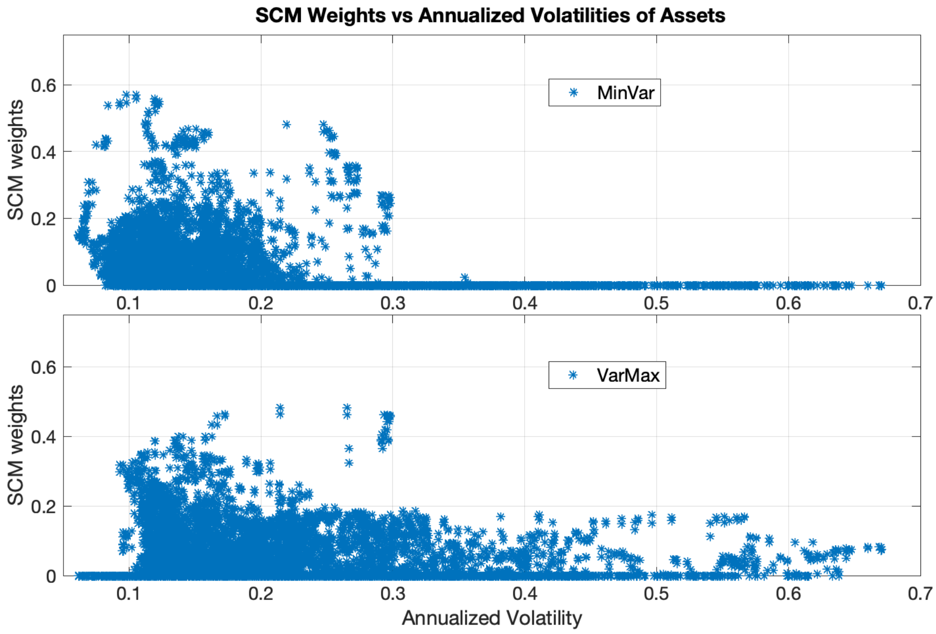

Fig. 4 VarMax and MinVar SCM weights versus the assets volatilities. As expected, MinVar weights are mostly non-zeros for the assets having the lowest volatilities. VarMax weights are more indifferent to the volatility levels.

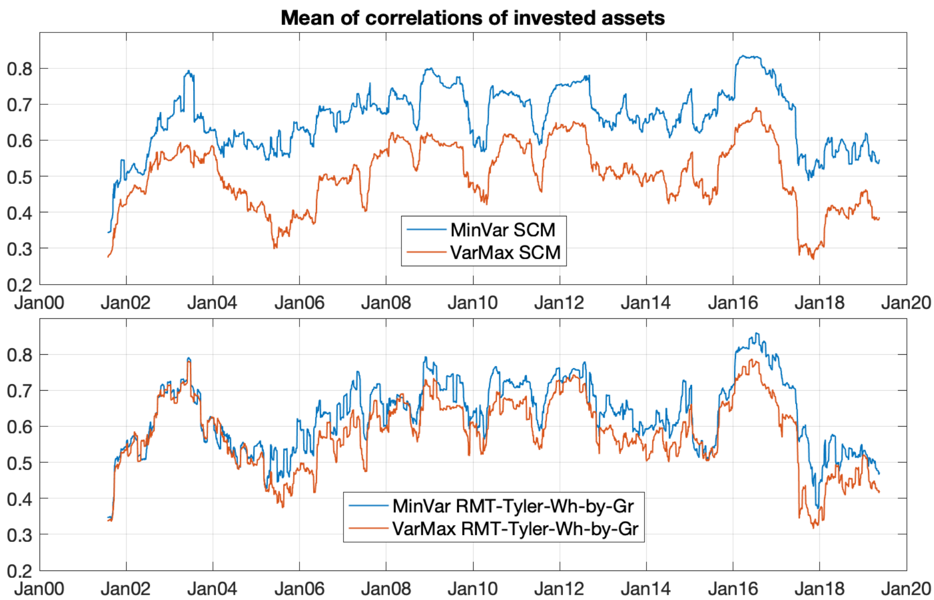

Fig. 5 Average correlation of the invested assets for the VarMax and MinVar portfolios combined with either SCM or RMT-Tyler-Wh-by-Gr method. VarMax SCM weights are assigned to the less correlated assets if compared to the SCM MinVar weights and the difference is reduced in the RMT-Tyler-Wh-by-Gr case.

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