FALSE ALARM REGULATION FOR OFF-GRID TARGET DETECTION WITH THE MATCHED FILTER

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ABSTRACT

In the state-of-the-art, the Probability of False Alarm (P_{FA})-threshold relationship for the popular Matched Filter (MF) is often derived assuming that unknown non-linear parameters lie on a grid. However, these parameters vary continuously in practice. This is known as the off-grid case. In this article, an asymptotic P_{FA}-threshold relationship for the popular Matched Filter is derived in the off-grid case under complex white Gaussian noise hypothesis using expected Euler characteristics. This asymptotic relationship fits very well with corresponding Monte-Carlo trials in the moderate to low P_{FA} regime.

***Index Terms*—** Radar detection, Off-Grid, GLRT, Matched Filter, False Alarm Regulation

1. INTRODUCTION

Detection of signals embedded in noise is a classical problem encountered in many fields [1]. Signals that depend on unknown parameters are usually dealt with using a Generalized Likelihood Ratio Test (GLRT) where the unknown parameters are replaced with their Maximum Likelihood Estimates (MLE) in the Likelihood Ratio Test. However, analytical MLE solutions are not always available. In this scenario, those parameters are usually supposed to lie on a discrete set, called the grid. Generally, this grid is chosen so as to have statistically independent tests. However, the parameters of interest have no reason to fall precisely on the grid. Thus, the signals of interest will always be mismatched with the signals under test. This off-grid mismatch may induce a loss in performance. This is the case when the GLRT coincides with the well-known Matched Filter, which can present up to a 3dB loss per parameter dimension.

Detection under mismatched signal models has been addressed for mismatch lying in a cone [2], quadratically constrained mismatch [3], among other types of mismatch [4–6].

Whereas off-grid mismatch has been explored extensively in sparse reconstruction [7, 8] and direction of arrival estimation [9], it has received less attention in detection (see, for example [10, 11]).

The most natural and efficient way to address the off-grid issue is to consider the true GLRT where the unknown parameters lie on a continuous space instead of an unrealistic discrete grid. There are two main issues with this procedure: it can be computationally heavy and its theoretical test statistics depend on the maximum of a continuum of non-independent variables, which may be difficult to evaluate. In [11], we derived a P_{FA}-threshold relationship for the off-grid Normalized Matched Filter. In this paper, our contribution focuses on the case of the off-grid radar Matched Filter with respectively one (Doppler or angle) and two (both Doppler and angle) unknown parameters. Based on results on random Gaussian fields [12], we propose a new asymptotic P_{FA}-threshold relationship in the case of complex circular white Gaussian noise.

Section 2 presents the signal model and the off-grid GLRT detector analyzed in this paper. In Section 3, we derive the corresponding theoretical P_{FA}-threshold relationship and its validation with Monte-Carlo simulations in Section 4.

Notations: Matrices are in bold and capital, vectors in bold. For any matrix $\mathbf{A}$ or vector, $\mathbf{A}^T$ is the transpose of $\mathbf{A}$ and $\mathbf{A}^H$ is the Hermitian transpose of $\mathbf{A}$. $\mathbf{I}$ is the identity matrix and $\mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Gamma})$ is the circular complex Normal distribution of mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Gamma}$. $\otimes$ is the Kronecker product. $\mathbf{a}_r$ and $\mathbf{a}_i$ are the complex and imaginary parts of the complex vector $\mathbf{a}$.

2. PROBLEM FORMULATION

Our problem of interest is to detect a potential known signal $\mathbf{s}(\boldsymbol{\xi}) \in \mathbb{C}^N$ of unknown amplitude, depending on unknown parameter $\boldsymbol{\xi}$ and embedded in noise. Formally, this reduces to solving the following hypothesis testing problem:

$$\begin{cases} H_0 : \mathbf{r} = \mathbf{n}\,, \\ H_1 : \mathbf{r} = a\,\mathbf{s}(\boldsymbol{\xi}) + \mathbf{n}\,, \end{cases} \tag{1}$$

where $\mathbf{n} \in \mathbb{C}^N$ is the noise vector distributed according to a complex circular Gaussian distribution $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{\Gamma})$ with known covariance matrix $\boldsymbol{\Gamma}$ and $a \in \mathbb{C}$ is the complex amplitude of the target.

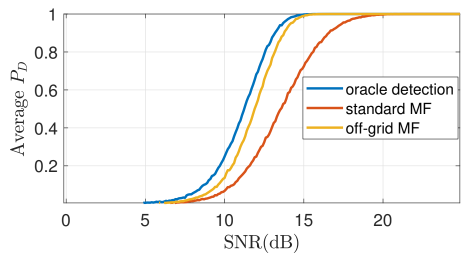


Fig. 1: STAP detection performance for off-grid targets drawn at random in a cell: (blue) oracle detection, (yellow) GLRT (6) and (red) Matched Filter (4). $P_{FA}=10^{-6}$, $N=8$, $P=4$.

In the rest of the paper, the number of unknown parameters will be fixed to one or two. In the case of a single parameter (e.g. Doppler shift $\boldsymbol{\xi}=\theta$ and N pulses), the signal $\mathbf{s}(\theta)$ can generally be expressed as:

$$\mathbf{s}(\theta)=\frac{1}{\sqrt{N}}\left[1, e^{2i\pi\theta}, \dots, e^{2i\pi(N-1)\theta}\right]^T \tag{2}$$

In the case of two unknown parameters ($\boldsymbol{\xi}=(\theta,\mu)^T$) as, for example, for Space-Time Adaptive Processing (STAP), the signal $\mathbf{s}(\theta,\mu)\in\mathbb{C}^{NP}$ can be represented by the joint Doppler θ (N pulses) and the direction of arrival μ (P sensors) parameters as:

$$\mathbf{s}(\boldsymbol{\xi})=\mathbf{s}(\theta)\otimes\mathbf{s}(\mu), \tag{3}$$

where $\mathbf{s}(\theta)\in\mathbb{C}^N$ and $\mathbf{s}(\mu)\in\mathbb{C}^P$ are defined as in (2).

When $\boldsymbol{\xi}$ is known and the complex amplitude a is unknown, the GLRT of this problem denoted as oracle GLRT reduces to the well-known MF test [13, 14]:

$$\frac{\left|\mathbf{s}(\boldsymbol{\xi})^H\boldsymbol{\Gamma}^{-1}\mathbf{r}\right|^2}{\mathbf{s}(\boldsymbol{\xi})^H\boldsymbol{\Gamma}^{-1}\mathbf{s}(\boldsymbol{\xi})} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} w^2. \tag{4}$$

In that case, the corresponding general P_{FA}-threshold relationship is given by the following equation:

$$P_{FA}=\exp\left(-w^2\right). \tag{5}$$

Since the true target parameters are unknown, tests are generally evaluated for parameter $\boldsymbol{\xi}$ values lying on a discrete grid. However, in reality, target parameters will rarely fall precisely on the grid. For typical Fourier resolution grid spacing, loss of the matched filter can reach up to 3dB per dimension (and thus up to 6dB in STAP), degrading detection performance, as illustrated in Figure 1.

The true GLRT solution, named *off-grid* GLRT or *off-grid* MF, is to evaluate (4) for all $\boldsymbol{\xi}$ varying continuously in the search domain $\mathcal{D}$, referred as a *cell* (interval $[k/N,(k+1)/N]$ in 1D) in the sequel:

$$\max_{\boldsymbol{\xi}\in\mathcal{D}}\frac{\left|\mathbf{s}(\boldsymbol{\xi})^H\boldsymbol{\Gamma}^{-1}\mathbf{r}\right|^2}{\mathbf{s}(\boldsymbol{\xi})^H\boldsymbol{\Gamma}^{-1}\mathbf{s}(\boldsymbol{\xi})} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} w^2. \tag{6}$$

The resulting empirical detector performance is shown in Figure 1. It is close to the oracle GLRT (which perfectly knows the position of targets) while the Matched Filter exhibits a detection loss due to the off-grid targets. To guarantee a given value of P_{FA}, the off-grid GLRT requires setting the threshold w^2 in (6). For that purpose, we need to know the statistics of the maximum of a non-independent random variable continuum. To our knowledge, there is no known solution to this problem in our context. The aim of the next section is to fill this gap in the white noise scenario ($\boldsymbol{\Gamma}=\mathbf{I}$).

3. AN ASYMPTOTIC P_{FA}-THRESHOLD RELATIONSHIP FOR THE OFF-GRID MATCHED FILTER

In this section, we derive a new asymptotic P_{FA}-threshold relationship for the off-grid MF under white noise, using expected Euler characteristics: studies [12,15] provide a general theoretical framework for solving this kind of problem.

3.1. Estimating the P_{FA} for a single parameter steering vector

In the case of range or Doppler detection, the MF response can be seen as the following 1D random field $Y(\theta)$:

$$Y(\theta)=\left|\mathbf{s}(\theta)^H\mathbf{r}\right|^2. \tag{7}$$

In order to use the results of [12], we need to introduce the following Gaussian random field:

$$X(\alpha,\theta)=\left(\boldsymbol{\gamma}_1(\theta)\cos\alpha+\boldsymbol{\gamma}_2(\theta)\sin\alpha\right)^T\underline{\mathbf{r}}, \tag{8}$$

where $\boldsymbol{\gamma}_1(\theta)=\begin{bmatrix}\mathbf{s}_r(\theta)\\ \mathbf{s}_i(\theta)\end{bmatrix}$, $\boldsymbol{\gamma}_2(\theta)=\begin{bmatrix}-\mathbf{s}_i(\theta)\\ \mathbf{s}_r(\theta)\end{bmatrix}$, $\underline{\mathbf{r}}=\begin{bmatrix}\mathbf{r}_r\\ \mathbf{r}_i\end{bmatrix}$ is a $2N$-real-valued noise vector following a centered Gaussian distribution of covariance $\mathbf{I}_{2N}/2$ and α is the phase of the product $\mathbf{s}(\theta)^H\mathbf{r}$ in $[0,2\pi]$. It can be shown [11] that $\max_{\alpha\in[0,2\pi]}X(\alpha,\theta)=\sqrt{Y(\theta)}$, so that the P_{FA} to characterize in this paper can be written as:

$$P_{FA}=\mathbb{P}\left(\max_{\theta\in\mathcal{D}}Y(\theta)>w^2\right)=\mathbb{P}\left(\max_{\alpha,\theta\in[0,2\pi]\times\mathcal{D}}X(\alpha,\theta)>w\right). \tag{9}$$

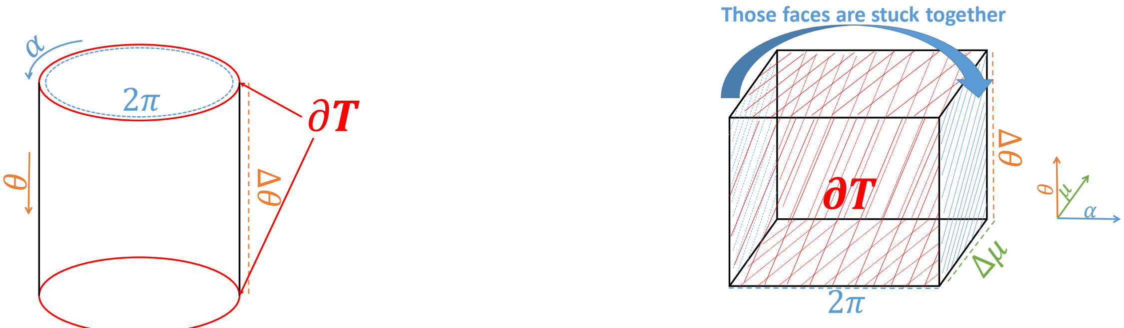
X is easily shown to be stationary and we denote by $\boldsymbol{\Lambda}$ the covariance matrix of its gradient $\left[\frac{\partial X}{\partial\alpha},\frac{\partial X}{\partial\theta}\right]$.

The *excursion set* $A_w(X)$ associated with X for a threshold w is defined as the set of parameters such that $X(\alpha,\theta)$ exceeds w [12]:

$$A_w(X)=\left\{(\alpha,\theta), X(\alpha,\theta)>w\right\}. \tag{10}$$

In [16], it is shown that $\mathbb{E}(\varphi(A_w(X)))$, the expected *Euler characteristic* of $A_w(X)$ is a precise estimation of the probability of $X(\alpha,\theta)$ exceeding w and so the desired PFA:

$$\left|P_{FA}-\mathbb{E}(\varphi(A_w(X)))\right|<O\left(\exp\left(-cw^2\right)\right), \tag{11}$$



(a) One unknown parameter: in this case, T is a hollow cylinder.

(b) Two unknown parameters: in this case, T can be viewed as a rectangular cuboid with two opposite faces stuck together.

Fig. 2: Representations of the search domains T.

for some $c > 1$. In this paper, we will not go into details about the Euler characteristic $\varphi(.)$ of a set: it is a topological characterization of the set, and for 2D sets, it counts the number of connected components minus the number of holes. For more details, readers can refer to the corresponding chapters of [16]. Our first result is based on the following fundamental theorem for random fields on $\mathbb{R}^2$:

Theorem 3.1 *[12] Let X be a zero-mean, isotropic stationary Gaussian random field on the parameter space $T \subset \mathbb{R}^2$ with gradient covariance matrix $\mathbf{\Lambda} = \lambda \mathbf{I}$ and assume that ∂T, defined as the boundary of T, is continuously differentiable except at an at most finite number of points. Then*

$$\mathbb{E}(\varphi(A_w(X))) = |T|\,\rho_2(w) + \frac{|\partial T|}{2}\rho_1(w) + \varphi(T)\Psi\left(\frac{w}{\sigma}\right), \quad (12)$$

where σ is the variance of X (constant since X is stationary), $\rho_k(w) = \dfrac{\exp(-w^2/2\sigma^2)\lambda^{k/2}}{(2\pi)^{(k+1)/2}\sigma^k} H_{k-1}\left(\dfrac{w}{\sigma}\right)$ with H_k the k-th Hermite polynomial and Ψ denotes the tail of the standard Gaussian distribution functions. $||$ denotes the volume.

Applying this general result to our specific problem, it can be shown that the following corollary holds:

Corollary 3.1.1 *The P_{FA}-threshold relationship of the off-grid MF tested on a cell $\mathcal{D} = [\theta_1, \theta_2]$ with steering vectors of size N is:*

$$\begin{aligned} P_{FA} \;=\; & \left(\sqrt{\frac{\pi(N^2-1)}{3}}(\theta_2-\theta_1)\,w + 1\right)\,\exp\left(-w^2\right) \\ & +\; O(\exp\left(-c\,w^2\right)) \qquad \text{for some } c > 1. \end{aligned} \quad (13)$$

When $\mathcal{D} = [0, 1]$, only the first term in the parenthesis should remain: this special case was given in [17] for Doppler detection.

As in the case of the off-grid NMF [11], the P_{FA} of the off-grid MF is expressed as the sum of a constant term, equal to the P_{FA} of the on-grid MF given in (5) and corresponding to the behavior of the random field at the boundaries of the search domain plus another term proportional to the length of the cell.

Let us give a sketch of the proof of the previous corollary. Our result is achieved when derivating the mean Euler characteristic of the excursion set of X defined in (10) and apply (11) in order to get the P_{FA}. First, note that in our case $\sigma^2 = 1/2$. The search domain for the target parameter θ is an interval $\mathcal{D} = [\theta_1, \theta_2]$. Since for all $\theta \in \mathcal{D}$, $X(0, \theta) = X(2\pi, \theta)$, T is shaped akin to an hollow cylinder in $\mathbb{R}^3$ as represented on Figure 2a: it has no boundaries along the α-axis. It results $|T| = 2\pi(\theta_2 - \theta_1)$, $|\partial T| = 4\pi$ and $\varphi(T) = 0$ since the Euler characteristic of a cylinder is zero. As the attentive reader may have noticed, our process X is not isotropic as required in the hypothesis of the theorem, but what matters in this hypothesis is that X should be stationary on ∂T and its gradient covariance matrix $\mathbf{\Lambda}$ should be proportional to identity matrix i.e. $\mathbf{\Lambda} = \lambda \mathbf{I}$. It is possible to have such a matrix $\mathbf{\Lambda}$ rewriting $\mathbf{s}$ in a centrosymmetric form and using an adequate change of variables $\nu = \pi\sqrt{\dfrac{N^2-1}{3}}\,\theta$.

Note that we could derive this relationship only because X is stationary under white noise. Unfortunately, this method cannot be applied to colored noise.

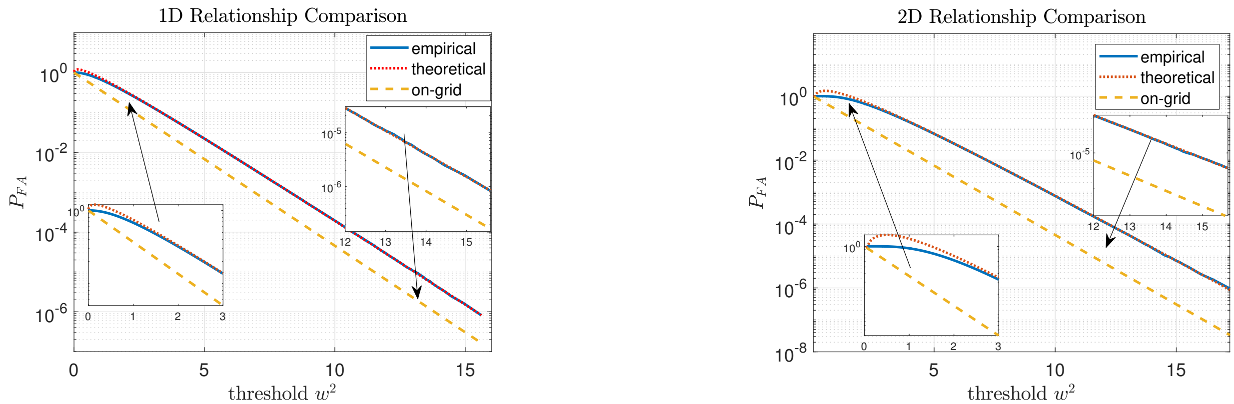
3.2. Estimating the P_{FA} for STAP detection

In the case of STAP detection, the MF test quantity can also be rewritten as a Gaussian random field, now depending on three unknown parameters (angle, Doppler shift, and phase):

$$X(\alpha, \boldsymbol{\xi}) = (\boldsymbol{\gamma}_1(\boldsymbol{\xi}) \cos\alpha + \boldsymbol{\gamma}_2(\boldsymbol{\xi}) \sin\alpha)^T\, \underline{\mathbf{r}}, \quad (14)$$

where the vector $\mathbf{s}$ used in $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ is now the steering vector for STAP detection defined in (3), and $\underline{\mathbf{r}}$ is a $2NP$ real-valued noise vector. This calls for the use of Theorem 3.3.2 in [12], in which the mean Euler characteristic of sufficiently "nice" random fields defined on a subset of $\mathbb{R}^3$ is given:

Theorem 3.2 *[12] Let X meet the same conditions as in theorem 3.1, except it is now defined on $T \subset \mathbb{R}^3$ and ∂T is twice*



(a) One unknown parameter. (b) Two unknown parameters.

Fig. 3: Theoretical relationship (13) (resp. (16)) (red) along an empirical (blue) obtained under white noise using $5\,10^7$ Monte Carlo draws with $N = 10$ (resp. $N = 8$ and $P = 4$). The on-grid relationship (5) (yellow) is shown for comparison purposes.

continuously differentiable except at edges or creases of finite length and vertices where the edges meet. Then:

$$\mathbb{E}(\varphi(A_w(X))) = |T|\,\rho_3(w) + \frac{|\partial T|}{2}\rho_2(w) + \frac{H(\partial T)}{\pi}\rho_1(w) + \varphi(T)\Psi\left(\frac{w}{\sigma}\right), \quad (15)$$

where $H(\partial T)$ is a curvature integral that equals the mean curvature if ∂T is smooth and involves all of the mean curvature, the lengths of the edges, and the angles between them otherwise: details are given in Lemma 8 of [18]. The other terms are defined as in Theorem 3.1.

Applying the previous theorem to $X(\alpha, \boldsymbol{\xi})$ leads to:

Corollary 3.2.1 *The P_{FA}-threshold relationship of the off-grid MF for STAP detection tested on a cell $\mathcal{D} = [\theta_1, \theta_2] \times [\mu_1, \mu_2]$ with steering vectors of size NP is:*

$$P_{FA} = \Big[\frac{\pi}{6}\Delta\theta\Delta\mu\sqrt{(N^2-1)(P^2-1)}\left(2w^2-1\right) + \sqrt{\frac{\pi}{3}}\left(\Delta\theta\sqrt{N^2-1} + \Delta\mu\sqrt{P^2-1}\right)w + 1\Big]e^{-w^2} + O(e^{-c\,w^2}) \qquad \text{for some } c > 1, \quad (16)$$

with $\Delta\theta = \theta_2 - \theta_1$ and $\Delta\mu = \mu_2 - \mu_1$.

Once again, when $\mathcal{D} = [0, 1] \times [0, 1]$, only the first term in the bracket remains.

Three terms appear in the developed formula: the first term accounts for the acceptance zone inside the surface of the 2D cell, while the second and third terms are the same as in (13). The second term translates what happens at the edge of the cells and is similar to what happens on a 1D cell. The corners of the cell explain the last term that is, once again, identical to the P_{FA} for the on-grid MF given in (5).

To give an idea of the proof, the shape of the parameter space is a bit more complicated in this case. The search domain for $\boldsymbol{\xi}$ is a 2D cell $\mathcal{D} = [\theta_1, \theta_2] \times [\mu_1, \mu_2]$. As X is still periodic in α, T can be viewed as a rectangular cuboid of dimensions 2π, $\Delta\theta$, $\Delta\mu$ where its faces along the phase axis are stuck together, as drawn in Figure 2b. This leads to $|T| = 2\pi\Delta\theta\Delta\mu$ and $|\partial T| = 4\pi(\Delta\theta + \Delta\mu)$. The quantity $\frac{H(\partial T)}{\pi}$ is equal to the mean length of the four edges, i.e., 2π, and finally, we have again $\varphi(T) = 0$. The previous remark on the isotropy of X still holds here.

4. NUMERICAL RESULTS

In order to verify (13) (resp. (16)), we computed an empirical P_{FA}-threshold relationship with 5×10^7 Monte Carlo draws in the Fourier resolution cell $[0, 1/N]$ (resp. $[0, 1/N] \times [0, 1/P]$), with steering vectors of fixed size $N = 10$ (resp. $N = 8$ and $P = 4$). The results are shown on Figure 3a (resp. 3b). In both cases, the relationship seems to be very accurate, except for P_{FA} close to one where there is a noticeable mismatch.

5. CONCLUSION

In this work, we analyzed the statistics of the off-grid GLRT, which is more robust to the presence of off-grid targets than the classical Matched Filter. We have derived its P_{FA}-threshold relationship under white noise for Doppler or angle and joint Doppler-angle STAP radar detection, but the relation is valid for all applications dealing with Fourier steering vectors as can be widely encountered in spectral analysis.

The next natural step would be to derive a similar relationship for colored noise distribution. This, however, is more involved due to the non-stationarity of the corresponding random fields.

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