A Tutorial on the Estimation and Detection Theory with Applications to Radar

Jean-Philippe Ovarlez[1,2]

[1]SONDRA, CentraleSupélec, France
[2]ONERA, The French Aerospace Lab, DEMR/MATS, France

jeanphilippe.ovarlez@centralesupelec.fr
https://www.jeanphilippeovarlez.com

Radar Autumn School 2024 - 19 October 2024

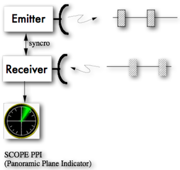
Background on Statistical Processing for Radar

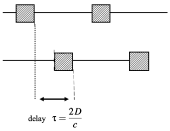
Contents

Outline

Range-Doppler Parameter Estimation - Range Measurement

Electromagnetic wave propagates with speed light c. The two-way propagation delay up to the distance D is $\tau = \frac{2\,D}{c}$





- Radar emitted signal: $s_e(t) = u(t)\,\exp\left(2 i\,\pi\,f_0\,t\right)$ where f_0 is the carrier frequency, and $u(.)$ the baseband signal,
- Radar received signal: $s_r(t) = \alpha\,s_e(t-\tau) + b(t)$ where α is the backscattering amplitude of the target and $b(.)$ is an additive noise.

$$s_r(t) = \alpha\,s_e\left(t - \frac{2\,D}{c}\right) + b(t)\,.$$

Range-Doppler Parameter Estimation - Velocity Measurement

Let us consider an illuminated moving target located for time t at range $D(t) = D_0 + v\,t$ where v is the radial target velocity.

If $\tau(t)$ is the two-way delay of the received signal at time t, the signal has been reflected at time $t - \tau(t)/2$ and the range $D(t)$ has to verify the following equation:

$$c\,\tau(t) = 2\,D\left(t - \frac{\tau(t)}{2}\right).$$

We obtain $\tau(t) = 2\,\dfrac{D_0 + v\,t}{c + v}$ and the model relative to signal return is:

$$\boxed{s_r(t) = \alpha\,s_e\left(\frac{c-v}{c+v}\,t - \frac{2\,D_0}{c+v}\right) + b(t)\,.}$$

The moving target is characterized in the signal return by a time-shift-compression/dilation of the emitted signal: action of Affine Group.

Range-Doppler Parameter Estimation - Velocity Measurement

Under the so-called *narrow-band* assumptions:

- $f_0 >> B$, where B is the bandwidth of baseband signal $u(.)$,
- $v << c$,
- $2\, B\, T << c/v$,

We have:
$$\begin{aligned} s_r(t) &= \alpha\, s_e\left(\frac{c-v}{c+v}\, t - \frac{2\, D_0}{c+v}\right) + b(t)\,, \\ &= \alpha\, \exp\left(i\, \phi\right) u\left(t - \frac{2\, D_0}{c}\right) \exp\left(2 i\, \pi\, f_0\, t\right) \exp\left(-2 i\, \pi\, \frac{2\, v}{c}\, f_0\, t\right) + b(t)\,. \end{aligned}$$

$$\boxed{s_r(t) = \alpha'\, s_e\left(t - \frac{2\, D_0}{c}\right) \exp\left(-2 i\, \pi\, f_d\, t\right) + b(t)\,.}$$

where $|\alpha'| = |\alpha|$ and where $f_d = \dfrac{2\, v}{c}\, f_0$ is called the Doppler frequency corresponding to moving target. The moving target is so characterized in the signal return by a time-shift/frequency shift of the emitted signal: action of Heisenberg Group .

Distance criterion - Ambiguity function and Matched Filter

One of the most important problem arising in radar theory is to separate targets in range and Doppler spaces. A $\mathcal{L}^2(\mathbb{R})$ distance R between two signals X and Y can be defined:

$$R^2 = \int_{-\infty}^{+\infty} |X(t) - Y(t)|^2 \, dt \,.$$

Minimizing this distance leads to maximize the inner product between X and Y (also known as Matched Filter):

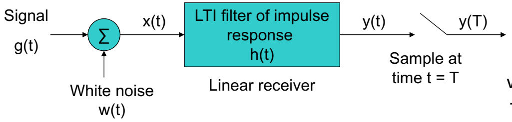
$$\int_{-\infty}^{+\infty} X(t)\, Y^*(t)\, dt \,.$$

According to the physical transformation of X, we obtain the so-called Ambiguity functions *[Woodward 53, Kelly 65]*:

- Example: $Y(t) = X(t-\tau)\, e^{2i\,\pi\,\nu\,t}$: $A(\tau,\nu) = \int_{-\infty}^{+\infty} X(t)\, X^*(t-\tau)\, e^{-2i\,\pi\,\nu\,t}\, dt \,,$
- Example: $Y(t) = \frac{1}{\sqrt{a}}\, X\left(a^{-1}\, t - b\right)$: $A(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} X(t)\, X^*\left(a^{-1}\, t - b\right)\, dt \,.$

Link with the so-called Matched Filter and Pulse Compression

Let us consider a linear time-invariant filter of impulse response $h(t)$. The filter input $x(t)$ consists of a pulse signal $g(t)$ corrupted by additive zero mean white noise $w(t)$ (with Power Spectral Density $\Phi_w(f) = N_0/2$). The output is $y(t) = g_0(t) + n(t)$, the signal and noise components of the input $x(t)$ for $0 \leq t \leq T$.



$$SNR = \frac{|g_0(T)|^2}{\sigma_n^2} = \frac{|g_0(T)|^2}{E\left[n^2(t)\right]},$$

where $|g_0(T)|^2$ is the power of the filtered signal $g(t)$ at $t = T$, and $\sigma_n^2 = E\left[n^2(t)\right]$ is the power of the filtered noise.

Since $|g_0(t)|^2 = \left|\int G(f)\, H(f) e^{j2\pi ft}\, df\right|^2$ and $\sigma_n^2 = R_n(0)$ where $R_n(\tau) = \int \frac{N_0}{2}\, |H(f)|^2\, e^{2i\pi f\tau}\, df$, the final expression for the output SNR is:

$$SNR = \frac{\left|\int H(f)\, G(f) e^{j2\pi fT}\, df\right|^2}{\frac{N_0}{2}\int |H(f)|^2\, df} \leq \frac{2}{N_0}\int |G(f)|^2\, df\,.$$

The SNR output is maximized only for the particular impulse response $h(t)$ that verifies:

$$H(f) = k\, G^*(f)\, e^{-2i\pi fT}\,, \forall k \in \mathbb{C}, \text{ or } h(t) = k\, g^*(T-t)\,.$$

Range resolution

Let us suppose N targets with amplitude $\{\alpha_i\}_{i\in[1,N]}$ located in range space at distance $\left\{d_i = \dfrac{c\,\tau_i}{2}\right\}_{i\in[1,N]}$. The received signal $s_r(t)$ is:

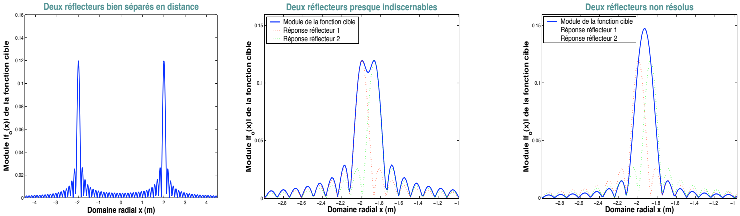
$$s_r(t) = \sum_{i=1}^{N} \alpha_i\, s_e(t-\tau_i) \overset{t\to f}{\Longrightarrow} S_r(f) = \sum_{i=1}^{N} \alpha_i\, S_e(f)\, e^{-2i\,\pi\,f\,\tau_i}\,.$$

The radar processing leads to evaluate for all τ, the following expression:

$$R(\tau) = \int_{-\infty}^{+\infty} s_r(t)\, s_e^*(t-\tau)\,dt \overset{t\to f}{\Longrightarrow} R(\tau) = \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{+\infty} |S_e(f)|^2\; e^{2i\,\pi\,f\,(\tau-\tau_i)}\,df\,.$$

- When $S_e(f) = 1$ for $f \in]-\infty,+\infty[$, $R(\tau) = \sum_{i=1}^{N} \alpha_i\,\delta(\tau-\tau_i)$,
- When $S_e(f) = 1$ for $f \in [-B/2,+B/2]$, $R(\tau) = \sum_{i=1}^{N} \alpha_i\, \dfrac{\sin\left(\pi\,B\,(\tau-\tau_i)\right)}{\pi\,B\,(\tau-\tau_i)}$.

Range resolution



(a) Distance réflecteurs : 4 m (b) Distance réflecteurs : 13 cm (c) Distance réflecteurs : 12.5 cm. (à la limite de résolution $\delta x = 12.5$ cm)

Figure: Here: $B = 1.2\,10^9$ Hz

The range resolution $\delta D = 0.125\,\mathrm{m}$ (defining the so-called *Range Bin*) is proportional to the inverse of the emitted signal bandwidth B:

$$\boxed{\delta D = \frac{c}{2}\,\frac{1}{B}\,.}$$

Velocity resolution

Let us suppose N targets with amplitude $\{\alpha_i\}_{i\in[1,N]}$ with Doppler $\left\{\nu_i = \dfrac{2\,v_i}{c}\,f_0\right\}_{i\in[1,N]}$. The received signal $S_r(f)$ is:

$$S_r(f) = \sum_{i=1}^{N} \alpha_i\, S_e(f-\nu_i) \overset{f\to t}{\Longrightarrow} s_r(t) = \sum_{i=1}^{N} \alpha_i\, s_e(t)\, e^{2i\,\pi\,\nu_i\,t}\,.$$

The radar processing leads to evaluate for all ν, the following expression:

$$R(\nu) = \int_{-\infty}^{+\infty} S_r(f)\, S_e^*(f-\nu)\, df \overset{t-f}{\Longrightarrow} R(\nu) = \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{+\infty} |s_e(t)|^2\, e^{-2i\,\pi\,t\,(\nu-\nu_i)}\, dt\,.$$

The velocity resolution δV (so-called *Doppler Bin*) is proportional to the inverse of the emitted signal duration (or integration time) T: $\boxed{\delta V = \dfrac{c}{2\,f_0}\,\dfrac{1}{T}\,.}$

Joint range and Velocity resolution

Let us suppose N targets with amplitude $\{\alpha_i\}_{i\in[1,N]}$ moving at velocity $\{v_i\}_{i\in[1,N]}$ and located in range space at distance $\left\{d_i = \dfrac{c\,\tau_i}{2}\right\}_{i\in[1,N]}$. The received signal $S_r(f)$ is:

$$s_r(t) = \sum_{i=1}^{N} \alpha_i\, s_e(t-\tau_i)\, e^{2i\,\pi\,\nu_i\,t}\,.$$

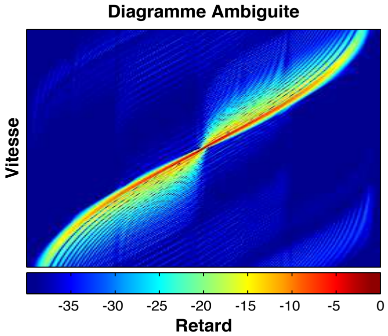
The radar processing (Matched Filter) leads to evaluate for all (τ,ν), the following expression:

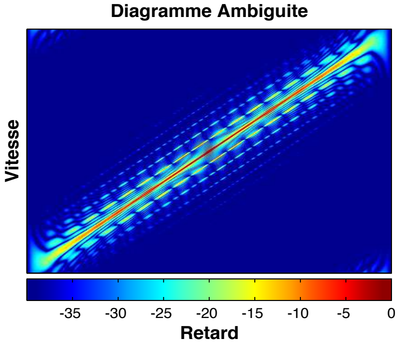
$$R(\tau,\nu) = \int_{-\infty}^{+\infty} s_r(t)\, s_e^*(t-\tau)\, e^{-2i\,\pi\,\nu\,t}\, dt\,.$$

This last equation is the superposition of the ambiguity functions *[Rihaczek 1969]* centered at $\{(\tau_i,\nu_i)\}_{i\in[1,N]}$

$$R(\tau,\nu) = \sum_{i=1}^{N} \alpha_i\, A(\tau-\tau_i, \nu-\nu_i)\,.$$

Some examples of Ambiguity Functions





- Best radar waveforms are those which look like a *thumbtack* form $(A(\tau, \nu) = \delta(\tau)\,\delta(\nu))$ but they definitely don't exist :-)
- Range and Doppler sidelobes can be troublesome for high density targets detection because of their superposition at different ranges and Doppler *[Rihaczek 1969]*.

Link with Minimal Bounds (Cramer Rao bounds)

- Let us define the second order moments (centered) of the signal $\sigma_t^2 = \int_{-\infty}^{+\infty} t^2 \, |s_e(t)|^2 \, dt \approx T^2$, $\sigma_f^2 = \int_{-\infty}^{+\infty} f^2 \, |S_e(f)|^2 \, df \approx B^2$ and the modulation index $m = \frac{-1}{2\pi} \, \mathrm{Im} \int_{-\infty}^{+\infty} t \, s_e(t) \, \frac{d s_e^*(t)}{dt} \, dt$. Under white Gaussian noise with variance σ^2, range and doppler accuracies are given by the following Cramer-Rao bounds *[Kay 93, Kay 98]*:

$$E\left[(\nu - \hat{\nu})^2\right] = \frac{\sigma^2}{4\,\pi^2\,\alpha^2} \frac{\sigma_f^2}{\sigma_f^2\,\sigma_t^2 - (m - t_0\,f_0)^2} \geq \frac{\sigma^2}{4\,\pi^2\,\alpha^2} \frac{1}{\sigma_t^2}\,, \tag{1}$$

$$E\left[(\tau - \hat{\tau})^2\right] = \frac{\sigma^2}{4\,\pi^2\,\alpha^2} \frac{\sigma_t^2}{\sigma_f^2\,\sigma_t^2 - (m - t_0\,f_0)^2} \geq \frac{\sigma^2}{4\,\pi^2\,\alpha^2} \frac{1}{\sigma_f^2}\,, \tag{2}$$

$$E\left[(\nu - \hat{\nu})(\tau - \hat{\tau})\right] = \frac{\sigma^2}{4\,\pi^2\,\alpha^2} \cdot \frac{m - t_0\,f_0}{\sigma_f^2\,\sigma_t^2 - (m - t_0\,f_0)^2} \tag{3}$$

- Radar uses to emit signal characterized with high time-bandwidth product $B\,T$.

Outline

Noise and Clutter in Radar

Thermal noise

Thermal noise for most radars corresponds to additive complex white Gaussian noise $\mathcal{CN}(\mathbf{0}_m, \mathbf{I}_m)$. This noise is generated by electronic devices in radar receivers.

What is the clutter?

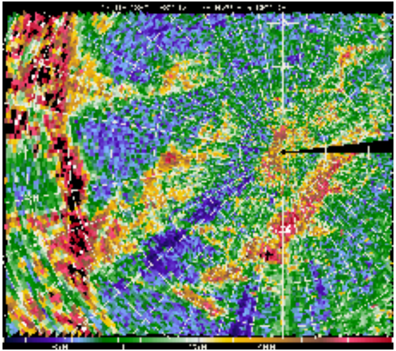
Clutter refers to radio frequency (RF) echoes returned from targets which are uninteresting to the radar operators and interfere with the observation of useful signals.
Such targets include natural objects such as ground, sea, precipitations (rain, snow or hail), sand storms, animals (especially birds), atmospheric turbulence, and other atmospheric effects, such as ionosphere reflections and meteor trails.
Clutter may also be returned from man-made objects such as buildings and, intentionally, by radar countermeasures such as chaff.

A statistical model for the clutter is necessary: in the following, we consider the clutter as a homogeneous Gaussian process!

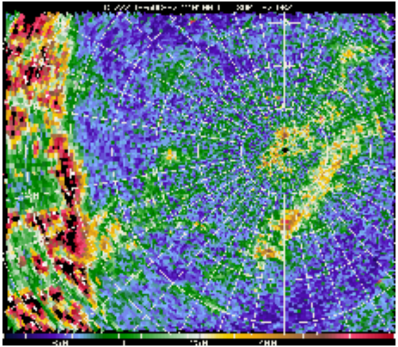
Noise and Clutter in Radar

Example of clutter map for different azimuth resolutions

resolution 3°



resolution 1°



Outline

Range-Doppler Radar Processing

- The cross-correlation operation is closely related to the so-called *Matched Filter* (filter which maximizes the SNR at its output). This is also known as the *pulse compression* processing. This matched filter offers the gain $B\,T$ on the noise power σ^2,
- The Doppler resolution is inversely proportional to the integration time. For monostatic radar (both emission and reception on the same antenna), radar prefers to cut off this long integration time into m pulses of duration T with Pulse Repetition Frequency (PRF) $F_r = 1/T_r$ (total integration time $m\,T_r$):

$$s(t) = \sum_{k=0}^{m-1} s_e(t - k\,T_r)\,.$$

Considering the signal return $s_r(t)$, the radar processing consists in evaluating:

$$R(\tau,\nu) = \int_{-\infty}^{+\infty} s_r(t)\, s^*(t-\tau)\, e^{-2i\,\pi\,\nu\,t}\, dt = \sum_{n=0}^{m-1} e^{-2i\,\pi\,\nu\,n\,T_r} \int_0^{T_r} s_r(u+n\,T_r)\, s_e^*(u-\tau)\, \cancel{e^{-2i\,\pi\,\nu\,u}}\, du\,.$$

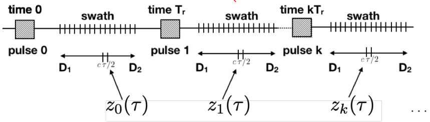
Neglecting the Doppler into the pulse duration leads to adapting the processing to the 0-Doppler: missing high-speed targets, bias in range estimation due to the ambiguity

Range-Doppler Radar Processing

When supposing non migrating target and neglecting the Doppler variation in the pulse, we can rewrite the processing as:

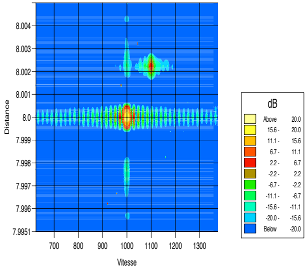
$$R(\tau,\nu) = \sum_{n=0}^{m-1} e^{-2i\,\pi\,\nu\,n\,T_r} \underbrace{\int_0^{T_r} s_r(u+\,n\,T_r+\tau)\,s_e^*(u)\,du}_{z_n(\tau)} = \mathbf{p}^H\,\mathbf{z}\,,$$

where $\mathbf{z} = (z_0(\tau), z_1(\tau), \ldots, z_{m-1}(\tau))^T$ and $\mathbf{p} = \left(1, e^{2i\,\pi\,\nu\,T_r}, \ldots, e^{2i\,\pi\,\nu\,(m-1)\,T_r}\right)^T$.



- For each range bin $c\,\tau/2$ (time T_r can be sampled at resolution $\delta\tau = 1/B$) on the range support $[D_1, D_2]$ of the analyzed swath, compute $z_n(\tau)$ corresponding to the time correlation between received signal and emitted pulse $s_e(t)$ at time $n\,T_r$,
- For each range bin $c\,\tau/2$, compute the Discrete Fourier Transform ($\mathbf{p}^H\,\mathbf{z}$) over the m coefficients $\{z_n(\tau)\}_{n\in[0,m-1]}$ to characterize Doppler spectrum in the spectral support $\nu \in [0, 1/T_r]$.

Range-Doppler Radar Processing



Example of the so-called Range-Doppler map of the processing data.

- Coherent Doppler processing brings an improvement of m on the Doppler resolution with regards to the one pulse processing $(\delta\nu = 1/(m\,T_r))$ as well as a gain m in SNR.
- Range resolution does not change. Always related by the pulse bandwidth,
- Appearance of the range ambiguities at ranges $c\,k\,T_r/2$, $k \in \mathbb{Z}$,
- Appearance of the Doppler ambiguities at Doppler frequency k/T_r, $k \in \mathbb{Z}$.

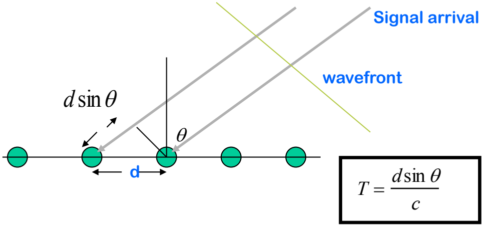
Outline

Array/Space-Time Adaptive Processing

Source locating in azimuth θ, at Doppler ν and in range bin $c\,\tau/2$

If the radar receives signal on antenna array, each antenna is collecting $s_r(t)$ delayed by the time shift $T = n\,d\,\sin\theta/c$ depending on its spatial position $n\,d$ ($n \in [0, N_s]$) on the array. Supposing that the array is non-dispersive $(N_s\,d\sin\theta << c/B)$, the concatenated $N_s \times m$-observation vector $\mathbf{y}$ collected by the radar on the antenna array for a given range bin $c\,\tau/2$ and Doppler ν is then:

$$\mathbf{y} = A\,\mathbf{p} \otimes \left(1, e^{2i\pi\,f_0\,d\,\sin\theta/c}, \ldots, e^{2i\pi\,f_0\,(N_s-1)\,d\,\sin\theta/c}\right)^T + \mathbf{b}\,.$$



Outline

General Formulation of All the Detection Problems

Set of two binary hypotheses

$$\begin{cases} H_0 : \mathbf{z} = \mathbf{b} \\ H_1 : \mathbf{z} = A\,\mathbf{p} + \mathbf{b} \end{cases}, \text{ where}$$

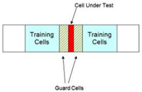
- $\mathbf{z}$ is a m-vector of data collected in a given measurement support. It can be range support, spatial support (Imaging), etc.
- The complex amplitude A of the target to detect is considered here deterministic (no fluctuation)
- The m-vector b represents the additive noise (thermal noise, photon noise, clutter, jam, etc.) characterized by a known (or unknown) PDF.
- The m-vector $\mathbf{p}$ represents the so-called deterministic *steering vector*: it can be relative to Doppler, Polarimetry, Interferomety, Wavelength, Spatial, time, joint Angular and Spectral information (STAP).

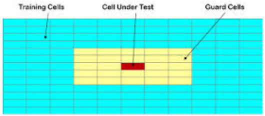
The problem here consists in choosing between H_1 hypothesis and H_0 hypothesis.

Problem Statement

- In a m-vector $\mathbf{z}$, detecting an unknown complex deterministic signal $\mathbf{s} = A\,\mathbf{p}$ embedded in an additive noise $\mathbf{y}$ can be written as the following statistical test:

$$\begin{cases} \text{Hypothesis } H_0: & \mathbf{z} = \mathbf{y} \quad \mathbf{z}_i = \mathbf{y}_i \quad i = 1, \ldots, n \\ \text{Hypothesis } H_1: & \mathbf{z} = \mathbf{s} + \mathbf{y} \quad \mathbf{z}_i = \mathbf{y}_i \quad i = 1, \ldots, n \end{cases}$$





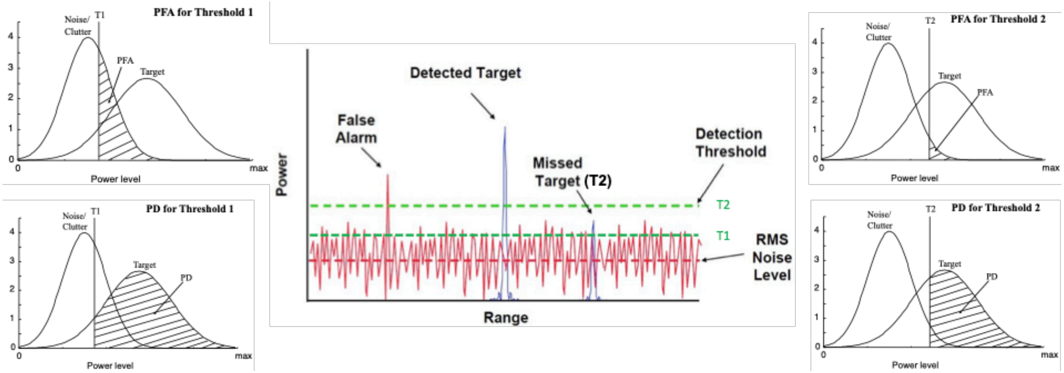
where the $\mathbf{z}_i$'s are n "signal-free" independent secondary data used to estimate the noise parameters. $\Rightarrow$ Neyman-Pearson criterion *[Kay 93, Kay 98]*

- Detection test: comparison between the Likelihood Ratio $\Lambda(\mathbf{z})$ and a detection threshold λ:

$$\Lambda(\mathbf{z}) = \frac{p_{\mathbf{z}}(\mathbf{z}/H_1)}{p_{\mathbf{z}}(\mathbf{z}/H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda\,,$$

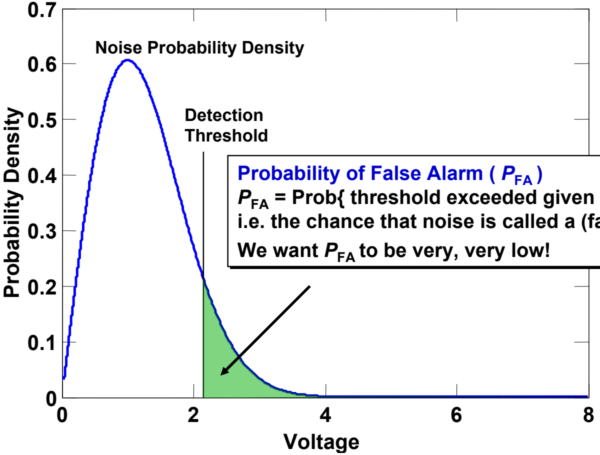
- Probability of False Alarm (type-I error): $P_{fa} = \mathbb{P}(\Lambda(\mathbf{z}) > \lambda/H_0)$
- Probability of Detection: $P_d = \mathbb{P}(\Lambda(\mathbf{z}) > \lambda/H_1)$ for different Signal-to-Noise Ratios (SNR),

Pd/Pfa



$$P_{fa} = \mathbb{P}(\Lambda(\mathbf{z}) > \lambda / H_0)\,, \qquad P_d = \mathbb{P}(\Lambda(\mathbf{z}) > \lambda / H_1)\,.$$

Pd/Pfa



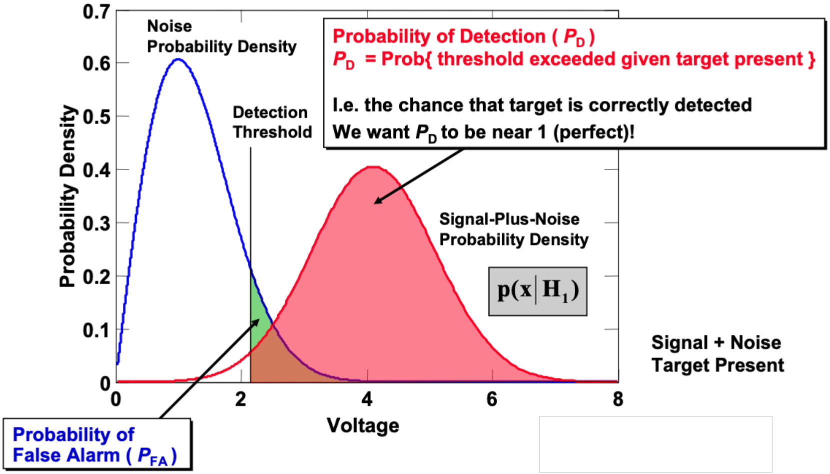
Noise Only
Target Absent

Probability of False Alarm (P_{FA})

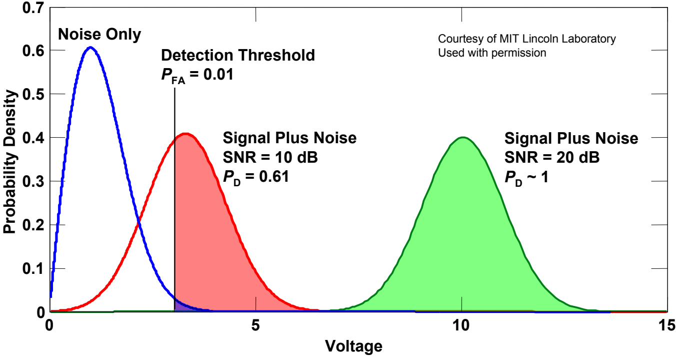
P_{FA} = Prob{ threshold exceeded given target absent }
i.e. the chance that noise is called a (false) target

We want P_{FA} to be very, very low!

Pd/Pfa

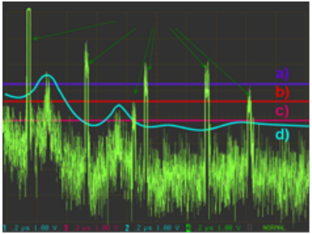

Noise
Probability Density
Detection
Threshold
Probability of Detection (P_D)
P_D = Prob{ threshold exceeded given target present }
I.e. the chance that target is correctly detected
We want P_D to be near 1 (perfect)!
Signal-Plus-Noise
Probability Density
$p(x|H_1)$
Signal + Noise
Target Present
Probability Density
Voltage
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
Probability of
False Alarm (P_{FA})

Pd/Pfa



- **P_D increases with target SNR for a fixed threshold (P_{FA})**
- **Raising threshold reduces false alarm rate and increases SNR required for a specified Probability of Detection**

False Alarm Regulation Importance



a. threshold is set too high: Probability of Detection = 20%
b. threshold is set optimal: Probability of Detection = 80%
   But one false alarm arises!
   False alarm rate = 1 / 666 = 1,5 $\cdot 10^{-3}$
c. threshold is set too low: a large number of false alarms arises!
d. threshold is set variable: constant false-alarm rate

CFAR Property

A detector is said Constant False Alarm Rate (CFAR property) if the PDF of the test is independent on the noise parameter (mean, covariance, variance, statistic) under H_0 hypothesis.

Real-Valued Multivariate Gaussian distribution

Definition

Let $\mathbf{x} = (x_1, \ldots, x_m)^T$ be a random vector. The vector $\mathbf{x}$ is Gaussian if and only if, for any sequence $\mathbf{a} = (a1, \ldots a_m)^T \in \mathcal{R}^m$ of real numbers, the scalar random variable $z = \mathbf{a}^T\,\mathbf{x} = \sum_{i=1}^{m} a_i\, x_i$ is a Gaussian variable.

We note $\boldsymbol{\mu} = \mathbb{E}(\mathbf{x})$ its mean and $\boldsymbol{\Sigma} = \mathbb{E}\left[(\mathbf{x}-\boldsymbol{\mu})\,(\mathbf{x}-\boldsymbol{\mu})^T\right]$ its covariance matrix.

Its PDF that is noted $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is given by

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\sqrt{(2\,\pi)^m\,|\boldsymbol{\Sigma}|}}\,\exp\left(-\frac{(\mathbf{x}-\boldsymbol{\mu})^T\,\boldsymbol{\Sigma}^{-1}\,(\mathbf{x}-\boldsymbol{\mu})}{2}\right).$$

Complex-Valued Multivariate Gaussian distribution

Definition

A random vector $\mathbf{z} = \mathbf{x} + j\mathbf{y}$ is complex Gaussian distributed $\mathbf{z} \sim \mathbb{C}\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}_z, \mathbf{P}_z)$ iif

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \Re(\boldsymbol{\mu}) \\ \Im(\boldsymbol{\mu}) \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_x & \boldsymbol{\Sigma}_{xy} \\ \boldsymbol{\Sigma}_{yx} & \boldsymbol{\Sigma}_y \end{bmatrix} \right)$$

with $\boldsymbol{\Sigma}_z = \mathbb{E}\left[(\mathbf{z}-\boldsymbol{\mu})(\mathbf{z}-\boldsymbol{\mu})^H\right] = \boldsymbol{\Sigma}_x + \boldsymbol{\Sigma}_y + j(\boldsymbol{\Sigma}_{yx} - \boldsymbol{\Sigma}_{xy})$ and
$\mathbf{P}_z = \mathbb{E}\left[(\mathbf{z}-\boldsymbol{\mu})(\mathbf{z}-\boldsymbol{\mu})^T\right] = \boldsymbol{\Sigma}_x - \boldsymbol{\Sigma}_y + j(\boldsymbol{\Sigma}_{yx} + \boldsymbol{\Sigma}_{xy})$

Circularity Property

$\mathbf{z} = \mathbf{x} + j\,\mathbf{y} \in \mathbb{C}^m$ is circularly symmetric $\mathbf{z} \sim \mathbb{C}\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}_z)$ iif $\mathbf{z} \overset{d}{=} e^{j\varphi}(\mathbf{z}-\boldsymbol{\mu})\ \forall \varphi \in [0; 2\pi[$.
Notably, $\boldsymbol{\Sigma}_x = \boldsymbol{\Sigma}_y$ and $\boldsymbol{\Sigma}_{yx} = \boldsymbol{\Sigma}_{xy} = \mathbf{0} \Leftrightarrow \boldsymbol{\Sigma}_z = 2\,\boldsymbol{\Sigma}_x$ and $\mathbf{P}_z = \mathbf{0}$.

$$p_{\mathbf{z}}(\mathbf{z}) = \frac{1}{\pi^m |\boldsymbol{\Sigma}_z|} \exp\left(-(\mathbf{z}-\boldsymbol{\mu})^H \boldsymbol{\Sigma}_z^{-1} (\mathbf{z}-\boldsymbol{\mu})\right)$$

Noise distribution

Central limit theorem

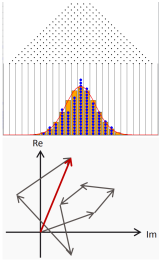
Let $x_1, x_2, \ldots, x_n$ be a sequence of random scalar i.i.d. variable with zero-mean and variance σ, then

$$\sqrt{n}\,\overline{x}_n \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} \mathcal{N}\left(0, \sigma^2\right) \quad \text{with} \quad \overline{x}_n = \frac{x_1 + x_2 + \ldots + x_n}{n}\,.$$

A Gaussian/Normal random variable has the largest entropy among all random variables of equal variance.

Scalar speckle noise (Goodman 1976)

$$z = \sum_{i=1}^{n} a_i \, \exp j\varphi_i \;\Rightarrow\; z \sim \mathbb{C}\mathcal{N}(0, \sigma^2),\; p(z) = \frac{1}{2\,\pi\,\sigma^2} \exp\left(-\frac{|z|^2}{2\,\sigma^2}\right).$$



The Galton board (top), Random walk (bottom)

This explains why the **Gaussian distribution** is often used to model the in-phase return of a large number of i.i.d. backscatterers in a radar resolution cell.

Link Between Covariance Matrix and Power Spectral Density 1/2

The Power Spectral Density $\Phi(f)$ characterizes, in a given range bin, the spectral (Doppler) fluctuations of a process $\mathbf{z} = (z_0, \dots, z_{m-1})^T$ collected from pulse to pulse.

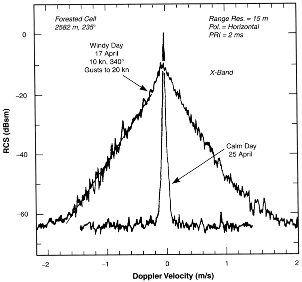


FIGURE 4.23 Power spectra of X-band radar returns from windblown trees.

[*Billingsley 1993*]

- Examples of some PSD models with $-1/(2\,T_r) \leq f \leq 1/(2\,T_r)$:

$$\Phi(f) = \Phi_0 \, \exp\left(-\frac{(f-f_c)^2}{2\,\sigma_f^2}\right), \quad \Phi(f) = \frac{\Phi_0}{1+\left(\frac{f}{f_c}\right)^n},$$

- Autocorrelation function (Wiener-Khintchine Theorem):

$$\rho(\tau) = \int_{-\infty}^{+\infty} \Phi(f)\, \exp\left(2 i \pi f \tau\right) df\,.$$

- Covariance Matrix: $\boldsymbol{\Sigma} = E\left[\mathbf{z}\,\mathbf{z}^H\right] = \begin{pmatrix} \rho(0) & \dots & \rho((m-1)\,T_r) \\ \vdots & \ddots & \vdots \\ \rho((m-1)\,T_r) & \dots & \rho(0) \end{pmatrix}.$

Link Between Covariance Matrix and Power Spectral Density 2/2

Examples of PSD and their associated covariance matrices:

- $\Phi(f) = N_0 \in \mathbb{R}^+$ that corresponds to a white noise leads to the CM equal to $\boldsymbol{\Sigma} = N_0\, B\, \mathbf{I}$, where B is the bandwidth of the receiver and $\mathbf{I}$ is the identity matrix.
- The exponential PSD $\Phi(f) = P_0\, \exp\left(-\alpha|f|\right)$, with $\alpha \in \mathbb{R}^+$ corresponds to the CM equal to $\left\{\rho(k\, Tr) = 2\,\alpha\, P_0\, \left(\alpha^2 + 4\pi^2 (k\, T_r)^2\right)^{-1/2}\right\}_{k\in[0,m-1]}$
- For any $0 \leq |\rho_0| \leq 1$, the practical covariance model $\boldsymbol{\Sigma}_{i,j} = \left\{\rho_0^{|i-j|}\right\}_{i,j\in[1,m-1]}$ leads to the PSD $|\Phi(f)| = \left|\dfrac{1 - \rho_0\, \exp\left(2 i \pi f m T_r\right)}{1 - \rho_0\, \exp\left(2 i \pi f T_r\right)}\right|$.
- Exercise: How to generate simulated (Matlab) random Gaussian vectors $\mathbf{z}$ with a given covariance matrix $\boldsymbol{\Sigma}_{i,j} = \left\{\rho_0^{|i-j|}\right\}_{i,j\in[1,m-1]}$?

General Detection Theory

When some parameters (noise, target) are unknown:

- GLRT Detection test: comparison between the Generalized Likelihood Ratio $\Lambda(\mathbf{z})$ and a detection threshold λ:

$$\Lambda(\mathbf{z}) = \frac{\max\limits_{\boldsymbol{\theta}} \max\limits_{\boldsymbol{\mu}} p_{\mathbf{z}/H_1}(\mathbf{z}, \boldsymbol{\theta}, \boldsymbol{\mu})}{\max\limits_{\boldsymbol{\mu}} p_{\mathbf{z}/H_0}(\mathbf{z}, \boldsymbol{\mu})} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda \,,$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\mu}$ represent respectively the unknown target parameter vector and the unknown noise parameter vector.

CFAR Property

A GLRT detector is said Constant False Alarm Rate (CFAR property) if the PDF of the GLRT test is independent on the noise parameter (mean, covariance, variance, statistic) under H_0 hypothesis.

General Estimation Theory: unknown deterministic parameters

- Maximum Likelihood Estimation (MLE) scheme: maximize the PDF with respect to the unknown parameter. Ex for noise parameter $\boldsymbol{\mu}$:

$$\widehat{\boldsymbol{\mu}} = \underset{\boldsymbol{\mu}}{\operatorname{argmax}}\, p_{\mathbf{z}/H_0}(\mathbf{z}, \boldsymbol{\mu})\,.$$

Example: Suppose n target-free i.i.d. m-vectors $\{\mathbf{z}_i\}_{i=1,n}$ where $\mathbf{z}_i \sim \mathcal{CN}\left(\mathbf{0}_m, \boldsymbol{\Sigma}\right)$ where $\boldsymbol{\Sigma}$ is an unknown covariance matrix. The MLE $\widehat{\mathbf{S}}_n$ is set by solving

$$\frac{\delta}{\delta\,\boldsymbol{\Sigma}} \log \prod_{i=1}^{n} p_{\mathbf{z}}(\mathbf{z}_i, \boldsymbol{\Sigma}) = \frac{\delta}{\delta\,\boldsymbol{\Sigma}^{-1}} \left(n\, \log \left|\boldsymbol{\Sigma}^{-1}\right| - \sum_{i=1}^{N} \mathbf{z}_i^H\, \boldsymbol{\Sigma}^{-1}\, \mathbf{z}_i \right) = \mathbf{0}\,.$$

Recalling that $\dfrac{\delta}{\delta\,\boldsymbol{\Sigma}^{-1}} \log \left|\boldsymbol{\Sigma}^{-1}\right| = \boldsymbol{\Sigma}^T$ and $\dfrac{\delta}{\delta\,\boldsymbol{\Sigma}^{-1}} \left(\mathbf{z}_i^H\, \boldsymbol{\Sigma}^{-1}\, \mathbf{z}_i\right) = \left(\mathbf{z}_i\, \mathbf{z}_i^H\right)^T$, we obtain:

Sample Covariance Matrix: MLE of the Gaussian problem

$$\widehat{\mathbf{S}}_n = \frac{1}{n} \sum_{i=1}^{n} \mathbf{z}_i\, \mathbf{z}_i^H\,.$$

Outline

Modeling Homogeneous Gaussian Noise/Clutter

Problem to solve in Gaussian environment

$$\begin{cases} H_0: & \mathbf{z} = \mathbf{y} & \mathbf{z}_i = \mathbf{y}_i & i = 1, \dots, n \\ H_1: & \mathbf{z} = \mathbf{s} + \mathbf{y} & \mathbf{z}_i = \mathbf{y}_i & i = 1, \dots, n \end{cases}$$

where $\mathbf{s} = A\,\mathbf{p}$, $\mathbf{y}$ and $\mathbf{y}_i \sim \mathcal{CN}\left(\mathbf{0}_m, \mathbf{\Sigma}\right)$, i.e. $p_{\mathbf{z}}(\mathbf{z}) = \dfrac{1}{\pi^m\,|\mathbf{\Sigma}|} \exp\left(-\mathbf{z}^H\,\mathbf{\Sigma}^{-1}\,\mathbf{z}\right)$

Goal: to choose the best hypothesis while minimizing the risk of being wrong (False Alarm) from an observation vector $\mathbf{z}$

$\Longrightarrow$ **All is known for Gaussian assumption!**

Sample Covariance Matrix (SCM)

When $\mathbf{\Sigma}$ is unknown, the Gaussian environment is modeled through the SCM: $\widehat{\mathbf{S}}_n = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \mathbf{z}_i\,\mathbf{z}_i^H$.

Properties of the SCM in homogeneous Gaussian noise/clutter environment

Properties of the SCM

- Simple Covariance Matrix estimator,
- Very tractable,
- Wishart distributed,
- Well-known statistical properties: unbiased and efficient.

$$\text{Then, } \sqrt{n}\,\text{vec}\left(\widehat{\mathbf{S}}_n - \mathbf{\Sigma}\right) \xrightarrow{d} \mathcal{CN}\left(\mathbf{0}_{m^2}, \mathbf{C}, \mathbf{P}\right),$$

$$\text{where} \quad \begin{array}{l} \mathbf{C} = \left(\mathbf{\Sigma}^* \otimes \mathbf{\Sigma}\right) \\ \mathbf{P} = \left(\mathbf{\Sigma}^* \otimes \mathbf{\Sigma}\right) \mathbf{K}_{m^2,m^2} \,. \end{array}$$

where $\mathbf{K}_{m,m}$ is the $m \times m$ commutation matrix transforming any m-vector $\text{vec}\left(\mathbf{A}\right)$ into $\text{vec}\left(\mathbf{A}^T\right)$.

Under Gaussian assumptions $\mathcal{CN}(\mathbf{0}_m, \mathbf{\Sigma})$, the Sample Covariance Matrix (SCM) is the most likely covariance matrix estimate (MLE) and is the empirical mean of the cross-correlation of n m-vectors $\mathbf{z}_k$:

$$\widehat{\mathbf{S}}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{z}_k\,\mathbf{z}_k^H\,.$$

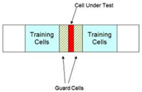
- This estimate is unbiased, efficient, Wishart distributed,
- n can represent any samples support called *the secondary data*: in time, spatial, angular domain, $\mathbf{z}_k$ a vector of any information collected in any domain:
  - in **Radar Detection**, it can represent the time returns collected in a given range bin of interest, n is here the range bin support
  - in **Array Processing**, it can represent the spatial information collected by the antenna array at a given time, n is here the time support,
  - in **Space Time Adaptive Processing**, it can represent the joint spatial and time information collected in a given range bin of interest, n is here the range bin support,
  - in **SAR** or **Hyperspectral imaging**, it can represent the polarimetric and/or interferometric, or spectral information collected for a given pixel of the spatial image, n is here the spatial support.

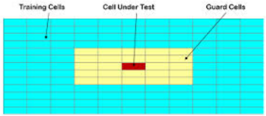
Outline

Example 0 - Detection Schemes in Range Doppler map

- In a scalar measurement z, detecting an unknown complex deterministic signal s embedded in an additive noise y can be written as the following statistical test:

$$\begin{cases} \text{Hypothesis } H_0: & z = y\,, & z_i = y_i & i = 1, \ldots, n \\ \text{Hypothesis } H_1: & z = s + y\,, & z_i = \mathbf{y}_i & i = 1, \ldots, n \end{cases}$$





where the z_i's are n "signal-free" independent secondary data used to estimate the noise parameters. $\Rightarrow$ Neyman-Pearson criterion *[Kay 93, Kay 98]*

Conventional detection framework on a mono-channel radar data mainly consists of locally comparing the complex amplitude of pixel z. In Gaussian homogeneous environment, i.e. $y \sim \mathcal{CN}(0, \sigma^2)$:

- Known power σ^2: global thresholding $\longrightarrow \Lambda(z) = |z|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$, leads to $\lambda = -\sigma^2 \log P_{fa}$,
- Unknown power σ^2: local thresholding $\longrightarrow \Lambda(z) = \dfrac{|z|^2}{\dfrac{1}{N}\displaystyle\sum_{k \neq i}^{N} |z_k|^2} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$ leads to $\lambda = N \left(P_{fa}^{-1/N} - 1\right)$.

The detection scheme only consists of thresholding the intensity of each map pixel.

Example 1 - Detection Schemes in Gaussian Noise

$$\text{Problem under study: } \begin{cases} \text{Hypothesis } H_0: & \mathbf{z} = \mathbf{b} \\ \text{Hypothesis } H_1: & \mathbf{z} = A\,\mathbf{p} + \mathbf{b}\,, \end{cases}$$

where $A \neq 0$ is a known complex scalar amplitude, $\mathbf{p}$ is the known steering vector and $\mathbf{b} \sim \mathcal{CN}(\mathbf{0}_m, \mathbf{\Sigma})$ with known covariance matrix $\mathbf{\Sigma}$. The probability density functions of the received m-vector $\mathbf{z}$ under each hypothesis are given by:

$$p_{\mathbf{z}/H_0}(\mathbf{z}) = \frac{1}{\pi^m\,|\mathbf{\Sigma}|}\exp\left(-\mathbf{z}^H\,\mathbf{\Sigma}^{-1}\mathbf{z}\right) \qquad p_{\mathbf{z}/H_1}(\mathbf{z}, A) = \frac{1}{\pi^m\,|\mathbf{\Sigma}|}\exp\left(-(\mathbf{z}-A\,\mathbf{p})^H\,\mathbf{\Sigma}^{-1}(\mathbf{z}-A\,\mathbf{p})\right).$$

The Log-Likelihood function $\log\dfrac{p_{\mathbf{z}/H_1}(\mathbf{z})}{p_{\mathbf{z}/H_0}(\mathbf{z})}$ can be simplified as: $\boxed{\Lambda(\mathbf{z}) = \mathbf{Re}\left(\mathbf{p}^H\,\mathbf{\Sigma}^{-1}\,\mathbf{z}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda\,.}$

The statistic of the test becomes:

$$\boxed{\Lambda(\mathbf{z}) \sim \mathcal{N}\left(0, \mathbf{p}^H\,\mathbf{\Sigma}^{-1}\,\mathbf{p}\right) \text{ under } H_0} \text{ and } \boxed{\Lambda(\mathbf{z}) \sim \mathcal{N}\left(\mathbf{Re}\left(A^H\,\mathbf{p}^H\,\mathbf{\Sigma}^{-1}\,\mathbf{p}\right), \mathbf{p}^H\,\mathbf{\Sigma}^{-1}\,\mathbf{p}\right) \text{ under } H_1}$$

Example 2 - Matched Filter (1)

$$\text{Problem under study: } \begin{cases} \text{Hypothesis } H_0: & \mathbf{z} = \mathbf{b}\,, \\ \text{Hypothesis } H_1: & \mathbf{z} = A\,\mathbf{p} + \mathbf{b}\,, \end{cases}$$

where A is unknown complex scalar amplitude, $\mathbf{p}$ is the known steering vector and $\mathbf{b} \sim \mathcal{CN}(\mathbf{0}_m, \boldsymbol{\Sigma})$ with known covariance matrix $\boldsymbol{\Sigma}$. The probability density functions of the received m-vector $\mathbf{z}$ under each hypothesis are given by:

$$p_{\mathbf{z}/H_0}(\mathbf{z}) = \frac{1}{\pi^m\,|\boldsymbol{\Sigma}|} \exp\left(-\mathbf{z}^H\,\boldsymbol{\Sigma}^{-1}\mathbf{z}\right), \quad p_{\mathbf{z}/H_1}(\mathbf{z}, A) = \frac{1}{\pi^m\,|\boldsymbol{\Sigma}|} \exp\left(-(\mathbf{z} - A\,\mathbf{p})^H\,\boldsymbol{\Sigma}^{-1}(\mathbf{z} - A\,\mathbf{p})\right).$$

Maximizing $p_{\mathbf{z}/H_1}(\mathbf{z}, A)$ with respect to A leads to the MLE $\hat{A}$: $\hat{A} = \dfrac{\mathbf{p}^H\,\boldsymbol{\Sigma}^{-1}\,\mathbf{z}}{\mathbf{p}^H\,\boldsymbol{\Sigma}^{-1}\,\mathbf{p}}$. Replacing it in the Log-Likelihood Ratio test, we obtain the well-known *Matched Filter*:

$$\Lambda_{MF}(\mathbf{z}) = \log \frac{\max_A p_{\mathbf{z}/H_1}(\mathbf{z}, A)}{p_{\mathbf{z}/H_0}(\mathbf{z})} = \frac{\left|\mathbf{p}^H\,\boldsymbol{\Sigma}^{-1}\,\mathbf{z}\right|^2}{\mathbf{p}^H\,\boldsymbol{\Sigma}^{-1}\,\mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda\,.$$

Example 2 - Matched Filter - Derivation of Performances (2)

Let $SNR = |A|^2\, \mathbf{p}^H\, \mathbf{\Sigma}^{-1}\, \mathbf{p}$ be the Signal to Noise Ratio of the target to be detected.

Under H_0 hypothesis, $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}_m, \mathbf{\Sigma})$ and $\Lambda_{MF}(\mathbf{z}) \sim \dfrac{1}{2}\chi^2(2)$. We have:

$$P_{fa} = \mathbb{P}\left(\Lambda_{MF}(\mathbf{z}) > \lambda_{MF}/H_0\right) = \int_{\lambda_{MF}}^{+\infty} e^{-u}\, du = \exp\left(-\lambda_{MF}\right),$$

$$\boxed{\lambda_{MF} = -\log P_{fa}\,.}$$

Under H_1 hypothesis, $\mathbf{z} \sim \mathcal{CN}(A\,\mathbf{p}, \mathbf{\Sigma})$ and $\Lambda_{MF}(\mathbf{z}, \hat{A}) \sim \dfrac{1}{2}\chi^2\left(2, 2\,SNR\right)$. We have:

$$\boxed{P_d = \mathbb{P}\left(\Lambda_{MF}(\mathbf{z}, \hat{A}) > \lambda_{MF}/H_1\right) = 1 - F_{\chi^2(2,\delta)}\left(2\,\lambda_{MF}\right)\,,}$$

where $F_{\chi^2(2,\delta)}(.)$ is the cumulative $\chi^2(2,\delta)$ density function with non-centrality parameter $\delta = 2\,SNR = 2\,A^2\,\mathbf{p}^H\,\mathbf{\Sigma}^{-1}\,\mathbf{p}$.

Example 3 - Normalized Matched Filter (1)

$$\text{Problem under study: } \begin{cases} \text{Hypothesis } H_0: & \mathbf{z} = \mathbf{b}\,, \\ \text{Hypothesis } H_1: & \mathbf{z} = A\,\mathbf{p} + \mathbf{b}\,, \end{cases}$$

where A is unknown complex scalar amplitude, $\mathbf{p}$ is the known steering vector and $\mathbf{b} \sim \mathcal{CN}(\mathbf{0}_m, \sigma^2\,\boldsymbol{\Sigma})$ with known covariance matrix $\boldsymbol{\Sigma}$ but unknown variance σ^2. The probability density functions of the received m-vector $\mathbf{z}$ under each hypothesis are given by:

$$p_{\mathbf{z}/H_0}(\mathbf{z}, \sigma^2) = \frac{1}{\pi^m\,\sigma^{2\,m}\,|\boldsymbol{\Sigma}|} \exp\left(-\frac{\mathbf{z}^H\,\boldsymbol{\Sigma}^{-1}\mathbf{z}}{\sigma^2}\right) \; p_{\mathbf{z}/H_1}(\mathbf{z}, A) = \frac{1}{\pi^m\,\sigma^{2\,m}\,|\boldsymbol{\Sigma}|} \exp\left(-\frac{(\mathbf{z} - A\,\mathbf{p})^H\,\boldsymbol{\Sigma}^{-1}(\mathbf{z} - A\,\mathbf{p})}{\sigma^2}\right).$$

- Maximizing $p_{\mathbf{z}/H_0}(\mathbf{z}, \sigma^2)$ with respect to σ^2 leads to the MLE: $\hat{\sigma}^2 = \dfrac{\mathbf{z}^H\,\boldsymbol{\Sigma}^{-1}\mathbf{z}}{m}$.
- Maximizing $p_{\mathbf{z}/H_1}(\mathbf{z}, \sigma^2, A)$ with respect to σ^2 and with respect to A leads to the MLEs:

$$\hat{\sigma}^2 = \frac{1}{m}\left(\mathbf{z}^H\,\boldsymbol{\Sigma}^{-1}\mathbf{z} - \frac{\left|\mathbf{p}^H\,\boldsymbol{\Sigma}^{-1}\,\mathbf{z}\right|^2}{\mathbf{p}^H\,\boldsymbol{\Sigma}^{-1}\,\mathbf{p}}\right) \text{ and } \widehat{A} = \frac{\mathbf{p}^H\,\boldsymbol{\Sigma}^{-1}\,\mathbf{z}}{\mathbf{p}^H\,\boldsymbol{\Sigma}^{-1}\,\mathbf{p}}\,.$$

Example 3 - Normalized Matched Filter (2)

Replacing it in the Log-Likelihood Ratio test, we obtain the well-known *Normalized Matched Filter*:

$$\Lambda_{NMF}(\mathbf{z}) = \log \frac{\max_{A} \max_{\sigma^2} p_{\mathbf{z}/H_1}(\mathbf{z}, \sigma^2, A)}{\max_{\sigma^2} p_{\mathbf{z}/H_0}(\mathbf{z}, \sigma^2)} = \frac{\left|\mathbf{p}^H \boldsymbol{\Sigma}^{-1} \mathbf{z}\right|^2}{\left(\mathbf{p}^H \boldsymbol{\Sigma}^{-1} \mathbf{p}\right) \left(\mathbf{z}^H \boldsymbol{\Sigma}^{-1} \mathbf{z}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{NMF} .$$

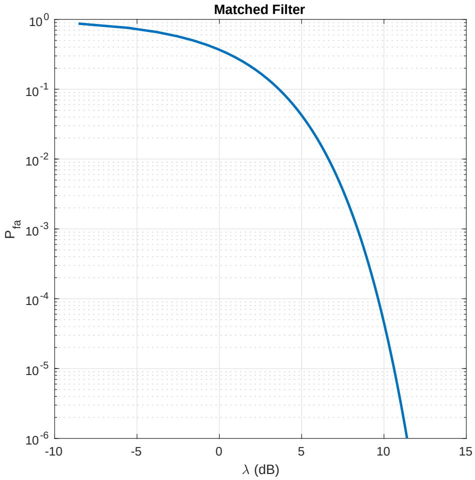
We can note that the NMF **is invariant with respect to a change scale** for $\mathbf{p}$, $\mathbf{z}$ or $\boldsymbol{\Sigma}$. Let $SNR = |A|^2 \, \mathbf{p}^H \boldsymbol{\Sigma}^{-1} \mathbf{p}$ be the Signal to Noise Ratio of the target to be detected. Under H_0 hypothesis, $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}_m, \sigma^2 \boldsymbol{\Sigma})$ and $\Lambda(\mathbf{z}) \sim \beta(1, m-1)$. We have:

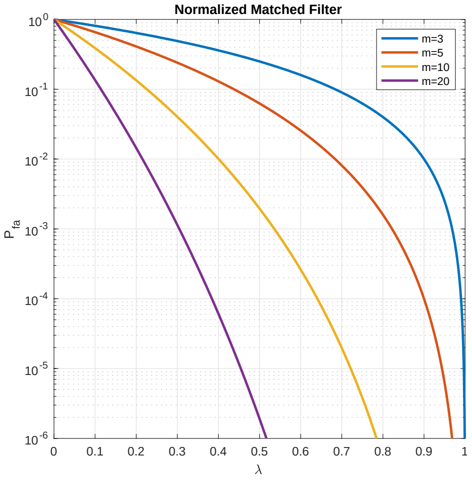
$$P_{fa} = \mathbb{P}\left(\Lambda_{NMF}(\mathbf{z}) > \lambda_{NMF}/H_0\right) = \left(1 - \lambda_{NMF}\right)^{m-1} ,$$

$$\lambda = 1 - P_{fa}^{1/(m-1)} .$$

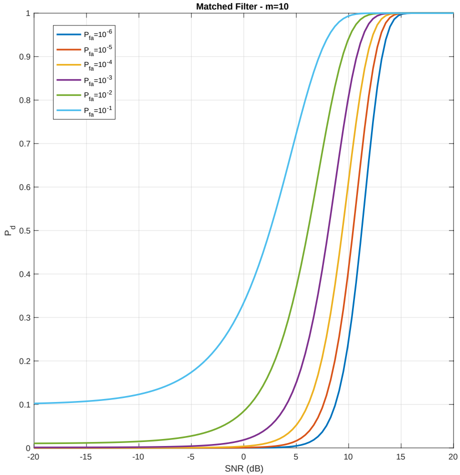
We can note that the threshold λ_{NMF} does not depend on unknown variance σ^2. The test is CFAR under H_0 hypothesis.

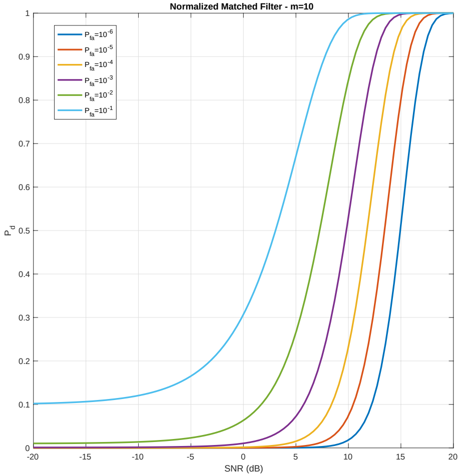
MF and NMF False Alarm regulation





MF and NMF Probability of Detection





Example 4 - Kelly and Adaptive Matched Filter (1)

Problem under study: $\begin{cases} \text{Hypothesis } H_0: & \mathbf{z} = \mathbf{b}\,, & \mathbf{z}_i = \mathbf{b}_i\,, \quad i = 1, \dots, n\,, \\ \text{Hypothesis } H_1: & \mathbf{z} = A\,\mathbf{p} + \mathbf{b}\,, & \mathbf{z}_i = \mathbf{b}_i\,, \quad i = 1, \dots, n\,. \end{cases}$

where the $\mathbf{z}_i$'s are n "signal-free" independent secondary data used to estimate the noise parameters, where A is unknown complex scalar amplitude, $\mathbf{p}$ is the known steering vector and $\mathbf{b} \sim \mathcal{CN}(\mathbf{0}_m, \boldsymbol{\Sigma})$ with unknown covariance matrix $\boldsymbol{\Sigma}$. The probability density function of the received m-vector $\mathbf{z}$ under hypothesis H_0 is given by:

$$p_{\mathbf{z},\{\mathbf{z}_k\}_k,\boldsymbol{\Sigma}/H_0}(\mathbf{z}) = \frac{1}{\pi^{m\,(n+1)}\,|\boldsymbol{\Sigma}|^{n+1}}\,\exp\left(-\mathbf{Tr}\left(\boldsymbol{\Sigma}^{-1}\left(\mathbf{z}\,\mathbf{z}^H + \sum_{k=1}^{n}\mathbf{z}_k\,\mathbf{z}_k^H\right)\right)\right).$$

With formulas $\dfrac{\delta\,\log|\boldsymbol{\Sigma}^{-1}|}{\delta\,\boldsymbol{\Sigma}^{-1}} = \boldsymbol{\Sigma}^T$ and $\dfrac{\delta\,\mathbf{tr}\left(\boldsymbol{\Sigma}^{-1}\,\mathbf{B}\right)}{\delta\,\boldsymbol{\Sigma}^{-1}} = \mathbf{B}^T$, we obtain:

$$\underset{\boldsymbol{\Sigma}}{\mathrm{argmax}}\; p_{\mathbf{z},\{\mathbf{z}_k\}_k,\boldsymbol{\Sigma}/H_0}(\mathbf{z}) = \frac{1}{n+1}\left(\mathbf{z}\,\mathbf{z}^H + \sum_{k=1}^{n}\mathbf{z}_k\,\mathbf{z}_k^H\right).$$

Example 4 - Kelly and Adaptive Matched Filter (2)

The probability density function of the received m-vector $\mathbf{z}$ under hypothesis H_1 is given by:

$$p_{\mathbf{z},\{\mathbf{z}_k\}_k,\mathbf{\Sigma},A/H_1}(\mathbf{z}) = \frac{1}{\pi^{m\,(n+1)}\,|\mathbf{\Sigma}|^{n+1}}\exp\left(-\mathbf{Tr}\left(\mathbf{\Sigma}^{-1}\left((\mathbf{z}-A\,\mathbf{p})\,(\mathbf{z}-A\,\mathbf{p})^H+\sum_{k=1}^{n}\mathbf{z}_k\,\mathbf{z}_k^H\right)\right)\right).$$

By denoting $\mathbf{S}=\sum_{k=1}^{n}\mathbf{z}_k\,\mathbf{z}_k^H$, we obtain $\underset{\mathbf{\Sigma}}{\operatorname{argmax}}\, p_{\mathbf{z},\{\mathbf{z}_k\}_k,\mathbf{\Sigma},A/H_1}(\mathbf{z}) = \dfrac{(\mathbf{z}-A\,\mathbf{p})\,(\mathbf{z}-A\,\mathbf{p})^H+\mathbf{S}}{n+1}$
and replacing these two expressions in the Generalized Log Likekihood Gatio leads to:

$$\Lambda(\mathbf{z}) = \frac{\left|\mathbf{z}\,\mathbf{z}^H+\mathbf{S}\right|}{\min\limits_{A}\left|(\mathbf{z}-A\,\mathbf{p})\,(\mathbf{z}-A\,\mathbf{p})^H+\mathbf{S}\right|}\underset{H_0}{\overset{H_1}{\gtrless}}\lambda\,.$$

If we note $\mathbf{z}_s=\mathbf{S}^{-1/2}\,\mathbf{z}$ and $\mathbf{p}_s=\mathbf{S}^{-1/2}\,\mathbf{p}$, we have:

$$\left|(\mathbf{z}-A\,\mathbf{p})\,(\mathbf{z}-A\,\mathbf{p})^H+\mathbf{S}\right| = |\mathbf{S}|\,\left|(\mathbf{z}_s-A\,\mathbf{p}_s)\,(\mathbf{z}_s-A\,\mathbf{p}_s)^H+\mathbf{I}_m\right| = |\mathbf{S}|\,\left(\|\mathbf{z}_s-A\,\mathbf{p}_s\|^2+1\right)$$

and $\min\limits_{A}|\mathbf{S}|\,\left(\|\mathbf{z}_s-A\,\mathbf{p}_s\|^2+1\right) = |\mathbf{S}|\,\left(\left\|\mathbf{P}_{\mathbf{p}_s}^{\perp}\,\mathbf{z}_s\right\|^2+1\right)$ where $\mathbf{P}_{\mathbf{p}_s}^{\perp}=\mathbf{I}_m-\mathbf{p}_s\,\mathbf{p}_s^H/\mathbf{p}_s^H\,\mathbf{p}_s$.

Example 4 - Kelly and Adaptive Matched Filter (3)

We obtain the following Generalized Likelihood Ratio test, known as the so-called *Kelly*'s test *[Kelly 86]*:

$$\Lambda_{Kelly}(\mathbf{z}) = \frac{\left|\mathbf{p}^H\,\mathbf{S}^{-1}\,\mathbf{z}\right|^2}{(\mathbf{p}^H\,\mathbf{S}^{-1}\,\mathbf{p})\;(1+\mathbf{z}^H\,\mathbf{S}^{-1}\,\mathbf{z})} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{Kelly} \qquad \text{where} \qquad \mathbf{S} = \sum_{k=1}^{n} \mathbf{z}_k\,\mathbf{z}_k^H\,.$$

This detector has good properties but is often (usually) replaced by a simpler one (so-called *two-step*), the *Adaptive Matched Filter [Robey 92]*:

$$\Lambda_{AMF}(\mathbf{z}) = \frac{\left|\mathbf{p}^H\,\widehat{\mathbf{S}}_n^{-1}\,\mathbf{z}\right|^2}{\mathbf{p}^H\,\widehat{\mathbf{S}}_n^{-1}\,\mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{AMF} \qquad \text{where} \qquad \widehat{\mathbf{S}}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{z}_k\,\mathbf{z}_k^H\,.$$

The covariance matrix estimate $\widehat{\mathbf{S}}_n = \dfrac{1}{n}\mathbf{S}$ is the *empirical* covariance matrix of the secondary data $\{\mathbf{z}_k\}_{k\in[1,n]}$ and is called *Sample Covariance Matrix* estimate.

Example 5 - Adaptive Normalized Matched Filter (1)

Detection in quasi-homogeneous Gaussian Noise: Problem under study:

$$\begin{cases} \text{Hypothesis } H_0: & \mathbf{z} = \mathbf{b}\,, & \mathbf{z}_i = \mathbf{b}_i\,, & i = 1, \dots, n\,, \\ \text{Hypothesis } H_1: & \mathbf{z} = A\,\mathbf{p} + \mathbf{b}\,, & \mathbf{z}_i = \mathbf{b}_i\,, & i = 1, \dots, n\,, \end{cases}$$

where the $\mathbf{z}_i$'s are n "signal-free" independent secondary data used to estimate the noise parameters, where A is unknown complex scalar amplitude, $\mathbf{p}$ is the known steering vector, where $\mathbf{b}_i \sim \mathcal{CN}(\mathbf{0}_m, \mathbf{\Sigma})$ and $\mathbf{b} \sim \mathcal{CN}(\mathbf{0}_m, \sigma^2\,\mathbf{\Sigma})$ with unknown covariance matrix $\mathbf{\Sigma}$ and unknown variance σ^2. The PDF under each hypothesis is given by *[Bandiera 09]*:

$$p_{\mathbf{z},\{\mathbf{z}_k\}_k,\mathbf{\Sigma}/H_0}(\mathbf{z}) = \frac{1}{\pi^{m\,(n+1)}\,|\mathbf{\Sigma}|^{n+1}}\,\exp\left(-\mathbf{z}^H\,\mathbf{\Sigma}^{-1}\,\mathbf{z} + \sum_{k=1}^{n} \mathbf{z}_k^H\,\mathbf{\Sigma}^{-1}\mathbf{z}_k\right),$$

$$p_{\mathbf{z},\{\mathbf{z}_k\}_k,\mathbf{\Sigma},\sigma^2,A/H_1}(\mathbf{z}) = \frac{1}{\pi^{m\,(n+1)}\,\sigma^{2\,m}\,|\mathbf{\Sigma}|^{n+1}}\,\exp\left(-\frac{(\mathbf{z}-A\,\mathbf{p})^H\,\mathbf{\Sigma}^{-1}\,(\mathbf{z}-A\,\mathbf{p})}{\sigma^2} + \sum_{k=1}^{n} \mathbf{z}_k^H\,\mathbf{\Sigma}^{-1}\mathbf{z}_k\right).$$

Example 5 - Adaptive Normalized Matched Filter (2)

The corresponding detector *[Scharf 94, Kraut 99]* is homogeneous of degree 0 with the variables $\mathbf{p}$, $\widehat{\mathbf{S}}_n$ and $\mathbf{z}$ and is named *Adaptive Normalized Matched Filter (ANMF)*:

$$\Lambda_{ANMF}(\mathbf{z}) = \frac{\left|\mathbf{p}^H\,\widehat{\mathbf{S}}_n^{-1}\,\mathbf{z}\right|^2}{\left(\mathbf{p}^H\,\widehat{\mathbf{S}}_n^{-1}\,\mathbf{p}\right)\,\left(\mathbf{z}^H\,\widehat{\mathbf{S}}_n^{-1}\,\mathbf{z}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{ANMF} \quad \text{where} \quad \widehat{\mathbf{S}}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{z}_k\,\mathbf{z}_k^H\,.$$

ANMF and Cosine Estimate

This detector is often called a Cosine Estimator as it has the dimension of a cosine squared between the steering vector $\mathbf{p}$ and the observation $\mathbf{z}$:

$$\Lambda_{ANMF}(\mathbf{z}) = \cos^2\left(\widehat{\mathbf{p},\mathbf{z}}\right).$$

Unlike the AMF which characterizes the power of a scalar product, the ANMF measures an angle. It is so more sensible to a possible mismatch between $\mathbf{p}$ and $\mathbf{z}$ (*[P. Develter 23]*).

Example 6 - Persymmetric Adaptive Matched Filter (1)

Many applications can result in a clutter covariance matrix that exhibits **some particular structure**. For example, radars use symmetrically spaced linear arrays for spatial domain processing and symmetrically spaced pulse trains for temporal domain processing.

- In these systems, the clutter covariance matrix $\mathbf{\Sigma}$ has the persymmetric property:

$$\mathbf{\Sigma} = \mathbf{J}_m \, \mathbf{\Sigma}^* \, \mathbf{J}_m \, ,$$

  where $\mathbf{J}_m$ is the m-dimensional antidiagonal matrix having 1 as non-zero elements.
- The signal vector is also persymmetric, i.e. it satisfies: $\mathbf{p} = \mathbf{J}_m \, \mathbf{p}^*$.
- The persymmetric structure of $\mathbf{\Sigma}$ can be exploited to improve its estimation accuracy compared to the SCM.

Example 6 - Persymmetric Adaptive Matched Filter (2)

We can build a two-step AMF with the persymmetric Maximum Likelihood (ML) estimate of the clutter covariance matrix instead of the SCM. The problem under study is:

$$\begin{cases} \text{Hypothesis } H_0: & \mathbf{x} = \mathbf{T}\,\mathbf{z} = \mathbf{T}\,\mathbf{b}\,, & \mathbf{x}_i = \mathbf{T}\,\mathbf{z}_i = \mathbf{T}\,\mathbf{b}_i\,, \quad i = 1, \ldots, n\,, \\ \text{Hypothesis } H_1: & \mathbf{x} = \mathbf{T}\,\mathbf{z} = A\,\mathbf{T}\,\mathbf{p} + \mathbf{T}\,\mathbf{b}\,, & \mathbf{x}_i = \mathbf{T}\,\mathbf{z}_i = \mathbf{T}\,\mathbf{b}_i\,, \quad i = 1, \ldots, n\,, \end{cases}$$

where $\mathbf{T}$ is the unitary matrix defined by:

$$\mathbf{T} = \begin{cases} \dfrac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I}_{m/2} & \mathbf{J}_{m/2} \\ i\,\mathbf{I}_{m/2} & -i\,\mathbf{J}_{m/2} \end{pmatrix} & \text{for } m \text{ even} \\ \\ \dfrac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I}_{(m-1)/2} & 0 & \mathbf{J}_{(m-1)/2} \\ 0 & \sqrt{2} & 0 \\ i\,\mathbf{I}_{(m-1)/2} & 0 & -i\,\mathbf{J}_{(m-1)/2} \end{pmatrix} & \text{for } m \text{ odd.} \end{cases}$$

Through this unitary transformation, secondary data $\mathbf{x}_i \sim \mathcal{CN}\left(\mathbf{0}, \mathbf{R}\right)$ where $\mathbf{R} = \mathbf{T}\,\boldsymbol{\Sigma}\,\mathbf{T}^H$ is a real covariance matrix

Example 6 - Persymmetric Adaptive Matched Filter (3)

Let us now investigate the ML estimate of the ***real*** covariance matrix $\mathbf{R}$ from the n transformed secondary data $\mathbf{x}_k$. The ML estimate $\widehat{\mathbf{R}}$ of real matrix $\mathbf{R}$ is unbiased and is given by:

$$\widehat{\mathbf{R}} = \mathcal{R}e(\widehat{\mathbf{R}}_n)\,,$$

where $\mathcal{R}e(.)$ stands for the real part, and where:

$$\widehat{\mathbf{R}}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{x}_k\,\mathbf{x}_k^H = \mathbf{T}\,\widehat{\mathbf{S}}_n\,\mathbf{T}^H \text{ where } \widehat{\mathbf{S}}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{z}_k\,\mathbf{z}_k^H\,.$$

- $n\,\widehat{\mathbf{R}}$ is real Wishart distributed with $2n$ degrees of freedom with parameter $\dfrac{1}{2}\,\mathbf{R}$,
- This result could be retrieved by the COMET procedure!

Example 6 - Persymmetric Adaptive Matched Filter (4)

The distribution of this new detector under hypothesis H_0 can be derived. Replacing $\widehat{\mathbf{R}}$ in the AMF (two-step procedure) leads to the following detection test, called the P-AMF:

$$\Lambda_{PAMF} = \frac{\left|\mathbf{s}^T\,\widehat{\mathbf{R}}^{-1}\,\mathbf{x}\right|^2}{\mathbf{s}^T\,\widehat{\mathbf{R}}^{-1}\,\mathbf{s}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{PAMF}\,,$$

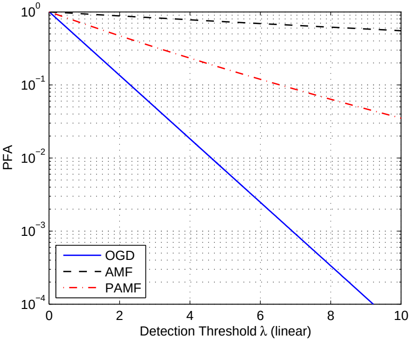
where $\mathbf{s} = \mathbf{T}\,\mathbf{p}$. In terms of the original data, we have, equivalently:

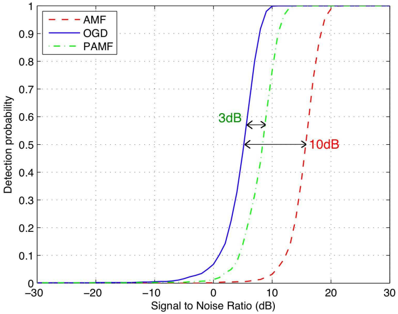
$$\Lambda_{PAMF} = \frac{\left|\mathbf{p}^H\,\mathbf{T}^H\left[\mathcal{R}e\left(\mathbf{T}\,\widehat{\mathbf{S}}_n\,\mathbf{T}^H\right)\right]^{-1}\mathbf{T}\,\mathbf{z}\right|^2}{\mathbf{p}^H\,\mathbf{T}^H\left[\mathcal{R}e\left(\mathbf{T}\,\widehat{\mathbf{S}}_n\,\mathbf{T}^H\right)\right]^{-1}\mathbf{T}\,\mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{PAMF}\,.$$

In the ML estimation procedure, taking into account the real structure of $\mathbf{R}$, or equivalently, the persymmetric structure of $\boldsymbol{\Sigma}$, virtually doubles the amount of secondary data.

Example 6 - Persymmetric Adaptive Matched Filter (5)

Theoretical λ/P_{fa} relationship: $P_{fa} = {}_2F_1\left(\frac{2n-m+1}{2}, \frac{2n-m+2}{2}, \frac{2n+1}{2}; -\frac{\lambda_{PAMF}}{n}\right)$.



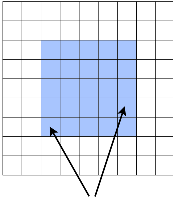


- Left figure: Threshold decreasing brought by the P-AMF compared to the AMF for $n = 25$ and $m = 20$.
- Right figure: Improvement of about 7dB in terms of detection for the PAMF compared to the AMF for this set of parameters.

Example 7 - Anomaly Detector (1)

Model : $\begin{cases} \text{Hypothesis } H_0 : \mathbf{x}_i = \mathbf{b}_i, & , \; i = 1, \dots, n \\ \text{Hypothesis } H_1 : \mathbf{x}_i = \alpha_i \, \mathbf{p} + \mathbf{b}_i, & , \; i = 1, \dots, n \end{cases}$ where $\mathbf{p}$, $\{\alpha_i\}_{i \in [1,n]}$ are unknown and $\{\mathbf{b}_i\}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{\Sigma})$. If we note $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^T$ and $\mathbf{X} = \begin{bmatrix} \mathbf{x}_1(1) & \dots & \mathbf{x}_n(1) \\ \vdots & \ddots & \vdots \\ \mathbf{x}_1(m) & \dots & \mathbf{x}_n(m) \end{bmatrix}$. The RXD GLRT test (Reed and Yu, 90) is defined as:

$$\Lambda_{RXD}(\mathbf{X}) = \frac{\left(\mathbf{X}\boldsymbol{\alpha}^T\right)^H \left(\mathbf{X}\mathbf{X}^H\right)^{-1} \left(\mathbf{X}\boldsymbol{\alpha}^T\right)}{\boldsymbol{\alpha}\boldsymbol{\alpha}^H}$$

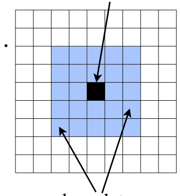


Taking a particular $\boldsymbol{\alpha} = [0, \dots, 0, 1, 0, \dots, 0]^T$, a more simple and well-known RXD version yields (the signal under test $\mathbf{x}_i$ is present in the covariance estimation!):

$$\Lambda_{RXD}(\mathbf{x}_i) = \mathbf{x}_i^H \, \hat{\mathbf{S}}_n^{-1} \, \mathbf{x}_i \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

Example 7 - Anomaly Detector (2)

Model: $\begin{cases} \mathcal{H}_0 : \mathbf{x} = \mathbf{b}, & , \ \{\mathbf{x}_i = \mathbf{b}_i\}_i \, , i = 1, \dots, n \\ \mathcal{H}_1 : \mathbf{x} = \alpha \, \mathbf{p} + \mathbf{b}, & , \ \{\mathbf{x}_i = \mathbf{b}_i\}_i \, , i = 1, \dots, n \end{cases}$ where $\{\mathbf{b}_i\}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{\Sigma})$, α and $\mathbf{p}$ are unknown. The Kelly GLRT test (Frontera, 14) is defined as:



$$\Lambda_{RXD}(\mathbf{x}) = \mathbf{x}^H \, \widehat{\mathbf{S}}_n^{-1} \, \mathbf{x} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

that corresponds to the Mahalanobis distance.

The Kelly test is Hotelling T^2 distributed: $\dfrac{n-m}{m\,(n+1)} RXD_{SCM}(\mathbf{c}/H_0) \sim F_{m,n-m}$.

Outline

Synthesis of CFAR Detection Schemes Under Gaussian Noise (1)

- Adaptive Matched Filter *[Robey 92]*:

$$\Lambda_{AMF}(\mathbf{z}) = \frac{\left|\mathbf{p}^H \, \widehat{\mathbf{S}}_n^{-1} \, \mathbf{z}\right|^2}{\mathbf{p}^H \, \widehat{\mathbf{S}}_n^{-1} \, \mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{AMF}$$

$$P_{fa} = {}_2F_1\left(n-m+1, n-m+2; n+1; -\frac{\lambda_{AMF}}{n}\right),$$

- Adaptive Kelly Filter *[Kelly 86]*:

$$\Lambda_{Kelly}(\mathbf{z}) = \frac{\left|\mathbf{p}^H \, \widehat{\mathbf{S}}_n^{-1} \, \mathbf{z}\right|^2}{\left(\mathbf{p}^H \, \widehat{\mathbf{S}}_n^{-1} \, \mathbf{p}\right) \left(n + \mathbf{z}^H \, \widehat{\mathbf{S}}_n^{-1} \, \mathbf{z}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{Kelly}$$

$$P_{fa} = \left(\frac{1}{\lambda_{Kelly}} - 1\right)^{n+1-m},$$

Synthesis of CFAR Detection Schemes Under Gaussian Noise (2)

- Adaptive Normalized Matched Filter *[Scharf 94, Kraut 99]*:

$$\Lambda_{ANMF}(\mathbf{z}) = \frac{\left|\mathbf{p}^H \widehat{\mathbf{S}}_n^{-1} \mathbf{z}\right|^2}{\left(\mathbf{p}^H \widehat{\mathbf{S}}_n^{-1} \mathbf{p}\right) \left(\mathbf{z}^H \widehat{\mathbf{S}}_n^{-1} \mathbf{z}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{ANMF} \;:$$

$$P_{fa} = (1-\lambda_{ANMF})^{n-m+1} \; {}_2F_1\left(n-m+2, n-m+1; n+1; \lambda_{ANMF}\right) .$$

- Persymmetric Adaptive Matched Filter *[Pailloux 09]*:

$$\Lambda_{PAMF} = \frac{\left|\mathbf{p}^H \mathbf{T}^H \left[\mathcal{R}e\left(\mathbf{T}\widehat{\mathbf{S}}_n \mathbf{T}^H\right)\right]^{-1} \mathbf{T}\,\mathbf{z}\right|^2}{\mathbf{p}^H \mathbf{T}^H \left[\mathcal{R}e\left(\mathbf{T}\widehat{\mathbf{S}}_n \mathbf{T}^H\right)\right]^{-1} \mathbf{T}\,\mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{PAMF} .$$

$$P_{fa} = {}_2F_1\left(\frac{2n-m+1}{2}, \frac{2n-m+2}{2}, \frac{2n+1}{2}; -\frac{\lambda_{PAMF}}{n}\right) .$$

The particular case of conventional Range Doppler 1/2

If we assume the noise if white Gaussian with known covariance matrix $\mathbf{\Sigma} = \sigma^2 \mathbf{I}$, then the conventional detection scheme $\Lambda_{MF}(\mathbf{z}) = \dfrac{\left|\mathbf{p}^H \mathbf{\Sigma}^{-1} \mathbf{z}\right|^2}{\mathbf{p}^H \mathbf{\Sigma}^{-1} \mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{MF}$ leads to the well known simplified test:

$$\Lambda_{MF}(\mathbf{z}) = \frac{\left|\mathbf{p}\,\mathbf{z}\right|^2}{\mathbf{p}^H \mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \lambda_{MF}\,,$$

The test consists, for each range bin, in comparing the normalized Discrete Fourier Transform of the vector $\mathbf{z}$ acquired for m pulses to a threshold. The corresponding PFA/threshold relationship is defined as:

$$\lambda_{MF} = -\sigma^2 \log P_{fa}\,.$$

The conventional Range Doppler algorithm makes implicitly assumption that the noise is white. In clutter environment, this processing is not optimal and we generally do not known the power σ^2 of the noise.

The particular case of conventional Range Doppler 2/2

If we assume the noise if Gaussian with unknown covariance matrix, then we have to use the conventional detection scheme $\Lambda_{AMF}(\mathbf{z}) = \dfrac{\left|\mathbf{p}^H\,\widehat{\mathbf{S}}_n^{-1}\,\mathbf{z}\right|^2}{\mathbf{p}^H\,\widehat{\mathbf{S}}_n^{-1}\,\mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_{AMF}$. For particular white noise with unknown power σ^2, we can build a *simplified* two-step detection scheme, assuming that $\widehat{\mathbf{S}}_n = \widehat{\sigma}^2\,\mathbf{I}$ where $\widehat{\sigma}^2 = \dfrac{1}{m}\displaystyle\sum_{k=1}^{m}\left|\mathbf{p}^H\mathbf{z}_k\right|^2$. The new detection test becomes:

$$\Lambda_{AMF}(\mathbf{z}) = |\mathbf{p}\,\mathbf{z}|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \widehat{\sigma}^2\,\lambda_{AMF}\,,$$

The test consists, for each range bin, in comparing the normalized Discrete Fourier Transform of the vector $\mathbf{z}$ acquired for m pulses to a adaptive threshold built with secondary data $\{\mathbf{z}_k\}_{k\in[1,m]}$. The corresponding PFA/threshold relationship is defined as:

$$\lambda_{AMF} = m\left(P_{fa}^{-1/m} - 1\right)\,.$$

This threshold tends to λ_{MF} for large value m of secondary data.

Outline

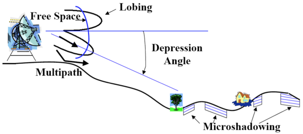
Examples of Gaussian Hypothesis Failure

High Resolution Radars

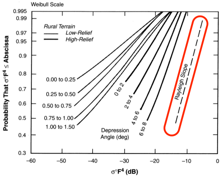
- Small number of scatterers in the cell under test - Varying number of scatterers from cell to cell - *Central Limit Theorem* non valid $\Rightarrow$ non-Gaussianity *[Jakeman 80]*
- No validity of conventional tools based on Gaussian statistics *[Farina 87, Gini 00, Jay 02]*.

Low-Grazing angles Illumination Radar

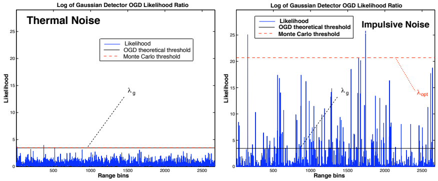
- Microshadowing $\Rightarrow$ impulsive clutter *[Billingsley 93]*
- Transitions of clutter areas, heterogeneity of spatial area under test $\Rightarrow$ difficulty to set up the detection test λ_{opt} and the Probability of False Alarm depending on the area.



Low-Grazing angle surveillance



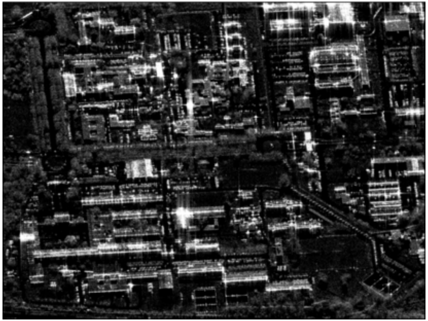
Non-Gaussian behavior



False Alarm regulation problem

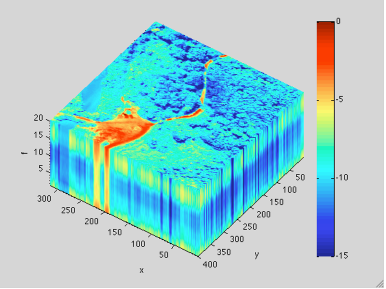
Please refer to *[F. Gini, A. Farina and M. S. Greco 2001]*

Examples of Gaussian Hypothesis Failure



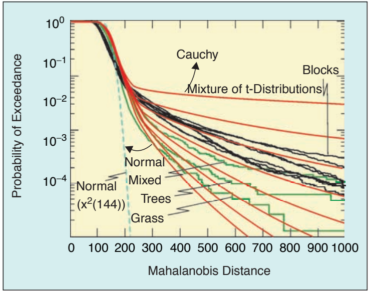
- The SAR images are more and more complex, detailed, heterogeneous. The spatial statistic of SAR images is not at all Gaussian,
- In polarimetry research field, almost all Non-Coherent Polarimetric Decomposition and classification techniques *[Lee 09, Formont 2012]* are generally based on conventional covariance matrix estimate (covariance or coherency matrix), typically the Sample Covariance Matrix (SCM),

Examples of Gaussian Hypothesis Failure



DSO data 2010

RXD CDF



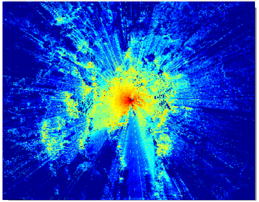
[*Manolakis 2002*]

Bad regulation of False Alarm rate for Anomaly Detector *[Reed 1990, Manolakis 2002, Ovarlez 2011, Frontera-Pons 2016]* and detectors of targets *[Frontera-Pons 2017]* in Hyperspectral Images when they are based on conventional SCM estimate.

Outline

Need of Better Approaches

Need to build alternatives to conventional approaches:



$\Rightarrow$ Better Covariance Matrix Estimation

Requirements:

- Background modeling: Compound Gaussian, SIRV (K-distribution, Weibull, etc.), CES (Multidimensional Generalized Gaussian Distributions, etc.),
- Estimation procedure: ML-based approaches, *M*-estimation, LS-based methods, etc.
- Adaptive detectors derivation and adaptive performance evaluation.

Some solutions will be proposed in Radar 2024 Tutorial on Robust Estimation and Detection Schemes in non-Standard Conditions for Radar, Array Processing and Imaging

End

Questions?

Bibliography

- F. Bandiera, D. Orlando, G. Ricci, Advanced radar detection schemes under mismatched signal models. Morgan & Claypool, 2009.
- P. Woodward, Probability and information theory with applications to radar. London (UK): Pergamon, 1953.
- E. J. Kelly, An adaptive detection algorithm, *Aerospace and Electronic Systems, IEEE Transactions on*, **23**(1):115–127, 1986.
- A. W. Rihaczek, Principles of high resolution radar. Mac-Graw-Hill, 1969.
- J. B. Billingsley, Ground Clutter Measurements for Surface-Sited Radar, *Technical Report*, 780, MIT, February 1993.
- J. Ward, Space-time adaptive processing for airborne radar. *IET Conference Proceedings*, (1), pp.2–2, January 1998.
- M. Soumekh, Fourier array imaging. Englewood Cliffs: Prentice Hall, 1994.
- M. Soumekh, Synthetic Aperture Radar signal processing with Matlab algorithms. New York: John Wiley and Sons, 1999.
- D. Mensa High resolution radar imaging. USA: Artech House, 1981.
- J. Bertrand, P. Bertrand and J.-P. Ovarlez, Frequency directivity scanning in laboratory radar imaging. International Journal of Imaging Systems and Technology, 5(1), 39-51, March 1994.
- J.-P. Ovarlez, G. Ginolhac and A. M. Atto, Multivariate linear time-frequency modeling and adaptive robust target detection in highly textured monovariate SAR image, *IEEE International Conference on Acoustics, Speech and Signal Processing* (ICASSP), pages 4029–4033, 2017.

Bibliography

- A. Farina, A. Russo and F. Scannapieco, Radar detection in coherent Weibull clutter, *Acoustics, Speech, and Signal Processing, IEEE Transactions on*, **35**(6), June 1987.
- P. Formont, *Statistical and Geometric Tools for the Classification of Highly Textured Polarimetric SAR Images*, Ph.D. Thesis, Paris Saclay University / ONERA, 2012.
- A. Mian, *Robust Change Detection in Time Series SAR Images*, PhD Thesis, Université de Paris Saclay / ONERA, 2019.
- A. Mian, J.-P. Ovarlez, A. M. Atto and G. Ginolhac, Design of New Wavelet Packets Adapted to High-Resolution SAR Images With an Application to Target Detection, *Geoscience and Remote Sensing, IEEE Transactions on*, **57**(6), pp.3919-3932, June 2019
- I. Reed and X. Yu, Adaptive multiple-band CFAR detection of an optical pattern with unknown spectral distribution, *Acoustics, Speech and Signal Processing, IEEE Transactions on*, **38**(10):1760–1770, 1990.
- D. Manolakis and G. Shaw, Detection algorithms for hyperspectral imaging applications. *IEEE Signal Processing Magazine*, **19**(1), pp.2943, Jan. 2002.
- J.-P. Ovarlez, S. K. Pang, F. Pascal, V. Achard and T.K. Ng, Robust Detection using the SIRV Background Modelling for Hyperspectral Imaging, *Proc IEEE-IGARSS*, Vancouver, Canada, July 2011.
- J. Frontera-Pons, *Robust Detection and Classification for Hyperspectral Imaging*, PhD Thesis, Paris Saclay University / ONERA, 2014.
- J. Frontera-Pons, M. A. Veganzones, S. Velasco-Forero, F. Pascal, J.-P. Ovarlez, J. Chanussot, Robust Anomaly Detection in Hyperspectral Imaging, *Proc. IEEE-IGARSS*, Quebec, Canada, July 2014.

Bibliography

- J. Frontera-Pons, M. A. Veganzones, F. Pascal and J.-P. Ovarlez, Hyperspectral Anomaly Detectors using Robust Estimators, *Selected Topics in Applied Earth Observation and Remote Sensing (IEEE-JSTARS), IEEE Journal of*, **9**(2), pp.720-731, 2016.
- J. Frontera-Pons, F. Pascal, J.-P. Ovarlez, Adaptive Nonzero-Mean Gaussian Detection, *Geoscience and Remote Sensing, IEEE Transactions on*, **55**(2), pp.1117-1124, 2017.
- J. Frontera-Pons, J.-P. Ovarlez and F. Pascal, Robust ANMF Detection in Noncentered Impulsive Background, *IEEE Signal Processing Letters*, **24**(12), pp.1891-1895, 2017.
- S. M. Kay. Fundamentals of Statistical Signal Processing: Estimation Theory, volume 1 of Prentice Hall Signal Processing Series. Prentice-Hall, Inc., 1993.
- S. M. Kay. Fundamentals of Statistical Signal Processing: Detection theory, volume 2 of Prentice Hall Signal Processing Series. Prentice-Hall PTR, 1998.
- F.C. Robey, D. R. Fuhrmann, E. J. Kelly and R. Nitzberg, A CFAR adaptive matched filter detector, *Aerospace and Electronic Systems, IEEE Transactions on*, **28**(1):208–216, 1992.
- E. J. Kelly, An adaptive detection algorithm. Aerospace and Electronic Systems, IEEE Transactions on, 23(1), 115-127, November 1986.
- F. Bandiera, D. Orlando, G. Ricci, Advanced radar detection schemes under mismatched signal models. Morgan & Claypool, 2009.
- L. L. Scharf and B. Friedlander, Matched subspace detectors, *Signal Processing, IEEE Transactions on*, **42**(8):2146–2157, 1994.

Bibliography

- S. Kraut and L. L. Scharf, The CFAR adaptive subspace detector is a scale-invariant GLRT, *Signal Processing, IEEE Transactions on*, **47**(9):2538–2541, 1999.
- A. Farina, A. Russo and F. Scannapieco, Radar detection in coherent Weibull clutter, *Acoustics, Speech, and Signal Processing, IEEE Transactions on*, **35**(6), June 1987.
- E. Jay, *Detection in non-Gaussian noise*, PhD thesis, University of Cergy-Pontoise / ONERA, France, 2002.
- F. Gini, M. Greco, M. Diani and L.Verrazzani, Performance analysis of two adaptive radar detectors against non-Gaussian real sea clutter data. *Aerospace and Electronic Systems, IEEE Transactions on*, **36**(4), 1429-1439, Oct. 2000.
- E. Jakeman, On the statistics of K-distributed noise. Journal of Physics A: Mathematical and General, **13**(1), 31, 1980.
- J.-S. Lee and E. Pottier, Polarimetric Radar Imaging, From Basics to Applications, CRC Press, 2009.
- P. Formont, *Statistical and Geometric Tools for the Classification of Highly Textured Polarimetric SAR Images*, PhD Thesis, Paris Saclay University / ONERA, 2012.
- F. Gini, A. Farina, M. S. Greco, Selected List of References on Radar Signal Processing, *Aerospace and Electronic Systems, IEEE Transactions on*, **37**(1), Jan. 2001