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# DIMENSIONALIZED WAVELET TRANSFORM WITH APPLICATION TO RADAR IMAGING

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## ABSTRACT

*Wavelet analysis is characterized by the constructive intervention of dilations and dilations in physics are associated with changes of measurement units. This double observation is at the basis of the introduction of the dimensionalized wavelet transform as a technique allowing to adapt the representation of dilations to the physical dimension of the field under study. An imaging process can then be built in accordance with the physical meaning required for the description. Lengthy general developments are avoided by illustrating the method with the practical problem of radar or sonar imaging of targets.*

## 1- INTRODUCTION

Practical study of communication signals requires the use of a clock as a measurement device but any choice of it remains free. This classical observation is at the basis of the general idea of equivalence between the signal interpretations delivered by various observers. Analytically, it justifies the interest for the affine group:

$$t \longrightarrow at + b \tag{1}$$

in the development of signal theory [1][2].

Working with the affine group (1) leads mathematically to introduce its linear representations on signals by transformations:

$$S(t) \longrightarrow S'(t) = a^r S(a^{-1}(t-b)) \tag{2}$$

where the parameter $r$ can be any real number. A dimensional interpretation of this parameter does exist but it depends on the characteristics of the measurement system we are using. As a matter of fact, every communication signal corresponds to the evolution of a physical quantity whose numerical values are functions of the chosen units of measurement. These units are generally interrelated, since an agreement between observers can always be introduced in order to ensure given values to some physical quantities like the velocity of light, the Planck constant, etc... Such a standardization procedure for the reference systems does not fix the relative spatial orientation of the axis or the relative position of the origin but it establishes connexions between the various scales. Hence, for example, the transformations of the fundamental units of time, length and mass can be taken of the form:

$$\begin{aligned} T &\longrightarrow aT \\ L &\longrightarrow aL \\ M &\longrightarrow a^{-1}M \end{aligned} \tag{3}$$

where $a$ is a positive dilational factor expressing the residual freedom of the observers. In this change of units any mechanical quantity is itself multiplied by some definite power of $a$ and this leads to a direct evaluation of the parameter occurring in (2).

The strict mathematical discussion of wavelet analysis [1][3] does not depend on the choice of a particular representation of the affine group and the classical choice $r = -\frac{1}{2}$ in equation (2) is mainly justified by the familiar form of the invariant scalar product it introduces. However, when practical applications are planned, the choice of $r$ has to be discussed. The object of the following developments is to show that such a discussion cannot be avoided without giving up the physical pertinence of the results.

Interpretation of wavelet analysis refers usually to an idea of localization which is neither trivial nor subjective in physics. In fact the essential of the notion is completely determined with the adoption of a measurement technique coherently defined for all observers. An illustration of this point is given below by deriving expressions for pure time-localized signals. The basic principle of observers equivalence implies that any signal $S_{t_0}$ localized at $t_0$ must be transformed into another localized signal $S_{at_0+b}$ in the clock change (1). By virtue of (2) this implies that $S_{t_0}$ verifies the functional equation:

$$S_{at_0+b}(t) = a^r S_{t_0}(a^{-1}(t-b)) \tag{4}$$

This equation is easily solved with the help of a Fourier transform defined by:

$$Z(f) = \int_R e^{-2i\pi ft} S(t)\, dt \tag{5}$$

and its solutions in the class of real signals of time are found to be of the form:

$$Z_{t_0}(f) = K e^{-2i\pi f t_0} |f|^{-r-1} \tag{6}$$



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where $K$ is a real constant. These functions are dependent on the parameter $r$ of the chosen representation, that is to say on the dimension of the physical localized signals they are representing. It can be noted that the "mathematical" localized expression:

$$S_{t_0}(t) = K\delta(t - t_0) \tag{7}$$

is only obtained for the particular case $r = -1$.

The above considerations open the way to a natural localization procedure founded on a careful comparison of the signal to study with localized signals of the same physical dimension. Practically they justify the introduction of the "dimensionalized" wavelet transform which is a mere rewriting of the classical one where the parameter $r$ in the generic relation (2) is left free. For consistency, the invariant scalar product:

$$(S_1, S_2) = \int_0^{\infty} Z_1(f) Z_2^*(f) f^{2r+1}\, df \tag{8}$$

has to be used in place of the classical one. This formulation does not particularize any observer and, whatever $r$, leads to a dimensionless wavelet coefficient. The isometry property of the transform ensures that the square modulus of this coefficient can be interpreted as a probability density on the time-frequency plane. This density presents itself as a particular case of $q$-dimensionalized distributions which transform according to:

$$R(t,f) \longrightarrow a^q R\big(a^{-1}(t-b), af\big) \tag{9}$$

in the clock change (1). Localized distributions $L_{t_0,f_0}$ on the time-frequency plane are naturally introduced through the correspondence:

$$L_{t_0,f_0} \longrightarrow L_{at_0+b, f_0/a}$$

expressing the equivalence of the observers connected by (1). Their analytic expression results from (9) and reads:

$$L_{t_0,f_0} = K f^q \delta(t - t_0)\delta(f - f_0) \tag{10}$$

where $K$ is any constant. Positive repartitions of such states give images characterized by a dimension which depends on the parameter $q$ in (10). In the dimensionalized wavelet imaging, the repartition is taken, up to a constant, equal to the square modulus of the wavelet coefficient. This procedure ensures that the ponderation is represented by a zero-dimensional density and that none of the observers is of special importance.

The above overview on wavelet imaging has stressed the importance of a dimensional study of the problem to solve previous to any application of the method. In this operation, each situation must be the object of a special attention and the most effective illustration of the technique is given by the development of an example. In the following the method is applied to the description of the reflectivity of radar (or sonar) targets whose backscattering coefficient is known. The general problem is presented in section II and the relevant dimensionalized wavelet transform is developed in section III. Dimensional arguments and resulting analytic expressions for the reflectivity are given in section IV.

## 2-RADAR IMAGING AND SIMILARITY GROUP.

The state of the art for radar or sonar imaging can be found in books and review papers [4][5][6]. In this section we give only a sketch of the technique in order to get a frame for the development of the wavelet approach of the problem.

Laboratory studies of radar targets make generally use of a monostatic coherent radar operating as shown on Fig.1. In a simple typical application the target rotates around a fixed axis perpendicular to the radar line of sight and polarization dependence is disregarded. In a more sophisticated experiment a positioning device can be used . The acquired data are values of the backscattering coefficient for different orientations of the target and different frequencies of irradiation. They realize the sampling of a complex function $H(f, \Omega)$ defined on a three-dimensional space where frequency $f$ and orientation $\Omega$ are spherical coordinates of a point. This complex function is sometimes called radar hologram. For each values of $f$ and $\Omega$, the square modulus of $H$ represents the radar cross section of the target. In usual three-dimensional scattering this quantity has the dimension of a surface and the function $H$ itself has the dimension of a length. For convenience the arguments of the function $H$ will be represented in the following by the vector $\mathbf{k}$ of modulus $2f/c$ and direction $\Omega$.

The idea of radar image is associated with the optical notion of reflectivity. It corresponds to a description of the electromagnetic target by a collection of bright points, each reflecting for a given frequency and for a given direction of illumination. The basic element of this model can thus be characterized by a point in a six-dimensional space $(\mathbf{x}, \mathbf{k})$ where $\mathbf{x}$ holds for the position of the point and $\mathbf{k}$ for its reflecting properties. As for the whole target, the direction of $\mathbf{k}$ corresponds to the working direction of the elementary reflector and $|\mathbf{k}|$ is related to its working frequency by:

$$f = \frac{c}{2} \mid \mathbf{k} \mid \tag{11}$$

For a given target, the expression of the function $H(\mathbf{k})$ depends on the reference system which is used. The study of

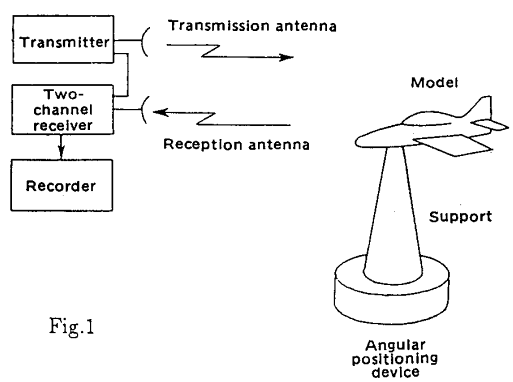


Fig.1

the transform of this function by change of observer constitutes the dimensional diagnosis of the problem.

The choice of a reference system concerns at first the origin of coordinates, the orientation of the axis and the scales of length and time. For two different choices, the corresponding coordinates of the bright points are related by a transformation which will be noted:

$$\mathbf{x} \longrightarrow \mathbf{x}' = a\mathcal{R}\mathbf{x} + \delta\mathbf{x} \qquad (12)$$

where $\mathcal{R}$ is a spatial rotation matrix, $a$ is a positive dilation factor and $\delta\mathbf{x}$ a translation vector.

The only physical parameter involved in the experiment is the velocity of light. All observers are supposed to agree on its value and this implies that their respective scales of length and time are always in the same ratio. This observation leads to complement the relation (12) by the transformation:

$$\mathbf{k} \longrightarrow \mathbf{k}' = a^{-1}\mathcal{R}\mathbf{k} \qquad (13)$$

since a frequency must transform as the inverse of a time.

In the actual experiment, each observer has also to perform a radar calibration by defining a center of phases. For practical reasons we will suppose that this point is systematically put at the origin of coordinates.

The above set of notations and conventions is sufficient to study the modification of the backscattering function in any change of observer. At first, we limit ourselves to the change (12)-(13) subject to the constraint $a \equiv 1$. The corresponding transformations of $H$ are directly obtained by classical arguments and read:

$$H \longrightarrow H' = e^{-2i\pi\mathbf{k}.\delta\mathbf{x}}H(\mathcal{R}^{-1}\mathbf{k}) \qquad (14)$$

It must be noted that the representation of the translations by an exponential factor in (14) results of the implicit assumption that the target size is small as compared to the radar-target distance. The change of length scale will not only affect the coordinates $(\mathbf{x}, \mathbf{k})$ of the bright points but also the unit used for the evaluation of $H$. With the requirement that $|H|^2$ be a surface we are led to the transformation law:

$$H(\mathbf{k}) \longrightarrow H'(\mathbf{k}) = ae^{-2i\pi\mathbf{k}.\delta\mathbf{x}}H(a\mathcal{R}^{-1}\mathbf{k}) \qquad (15)$$

Relation (15) presents itself as a representation of the three-dimensional similarity group. It connects observations of the same target for various reference systems but it can also be used to characterize the observations of similar targets by a unique observer. This latter remark is at the starting point of wavelets introduction in radar imaging.

## 3-APPEARANCE OF THE DIMENSIONALIZED WAVELET TRANSFORM.

Let $\phi(\mathbf{k})$ be the backscattering function of a reference target which is:

- located in the neighborhood of $\mathbf{x} = \mathbf{0}$
- reflecting essentially in the direction of the unitary vector $\mathbf{n}$ and at a frequency given by $|\mathbf{k}| = 2f/c = 1$
- invariant by rotations of axis $\mathbf{n}$.

By application of (15) to $\phi(\mathbf{k})$ it is possible to generate a family of functions $\{\phi_{\mathbf{x}_0,\mathbf{k}_0}(\mathbf{k}) \mid \mathbf{x}_0 \in R^3, \mathbf{k}_0 \in R^3_*\}$. Each element of this family is indexed by vectors $\mathbf{x}_0$, $\mathbf{k}_0$ which correspond respectively to vectors $\mathbf{x} = \mathbf{0}$ , $\mathbf{k} = \mathbf{n}$ by formulas (12) and (13). The constructing process ensures that the family is the same for all observers and depends only on the reference target. In fact we have obtained a set of coherent states for the similarity group which can be used for wavelet analysis [7]. The particular features of the present situation are that:

- the parameter space is the quotient of the similarity group by the little group (due to the rotational symmetry of the reference target)
- the choice of the exploited representation is founded on dimensional arguments.

The use of the family $\{\phi_{\mathbf{x}_0,\mathbf{k}_0}\}$ as a wavelet basis requires the introduction of an invariant scalar product which will be noted:

$$(H_1, H_2)_3 = \int_{S_2} d\Omega \int_0^\infty dk\, kH_1(k,\Omega)H_2^*(k,\Omega) \qquad (16)$$

where $k$ and $\Omega$ are the spherical coordinates for the vector $\mathbf{k}$. Adoption of (16) allows to introduce the wavelet coefficient of a function $H$ by the expression :

$$C_3(\mathbf{x}_0, \mathbf{k}_0) = (H, \phi_{\mathbf{x}_0,\mathbf{k}_0})_3 \qquad (17)$$

Straightforward computations show that this definition leads to reconstruction and isometry formulas given by:

$$H(\mathbf{k}) = [K_3(\phi)]^{-1} \int C_3(\mathbf{x}_0, \mathbf{k}_0)\phi_{\mathbf{x}_0,\mathbf{k}_0}(\mathbf{k})\, d\mathbf{x}_0\, d\mathbf{k}_0 \qquad (18)$$

and

$$\| H \|_3 = [K_3(\phi)]^{-1} \int | C_3(\mathbf{x}_0, \mathbf{k}_0 |^2\, d\mathbf{x}_0\, d\mathbf{k}_0 \qquad (19)$$

where $K_3$ characterizes the generating wavelet by:

$$K_3(\phi) = \int | \phi(\mathbf{k}) |^2 | \mathbf{k} |^{-4}\, d\mathbf{k} \qquad (20)$$

Formulas (17)-(20) are relative to the wavelet transformation for functions of the three-dimensional variable $\mathbf{k}$. However, in actual situations, we are not always working with the whole backscattering coefficient $H(\mathbf{k})$ but also with sections of it by an axis drawn from the origin or by a plane containing the origin. In these two problems, the analysis concerns functions of two or three variables whose transformations by changes of observers are given by restricted forms of (15).

The useful tool in the unidimensional case is a wavelet transform founded on the basis:

$$\phi_{x_0,k_0} = \frac{1}{k_0}e^{-2i\pi k x_0}\phi(\frac{k}{k_0}) \qquad (21)$$

This basis must be exploited with the invariant scalar product:

$$(H_1, H_2)_1 = \int_0^\infty dk\, kH_1(k)H_2^*(k) \qquad (22)$$

and the explicit study of the transformation exhibits the "admissibility" coefficient:

$$K_1(\phi) = \int_0^\infty | \phi(k) |^2 \ dk \tag{23}$$

In the two-dimensional problem, the wavelet basis is of the form:

$$\phi_{\mathbf{x}_0,\mathbf{k}_0}(\mathbf{k}) = \frac{1}{k_0} e^{-2i\pi \mathbf{k}.\mathbf{x}_0} \phi(\frac{k}{k_0}, \theta - \theta_0) \tag{24}$$

where $(k, \theta)$ and $(k_0, \theta_0)$ are polar coordinates of vectors $\mathbf{k}$ and $\mathbf{k}_0$ respectively. The relevant scalar product is the usual one given by:

$$(H_1, H_2) = \int_0^{2\pi} d\theta \int_0^\infty dk\, k H_1(k,\theta) H_2^*(k,\theta) \tag{25}$$

The development of the wavelet transformation using (24) introduces the admissibility coefficient:

$$K_2(\phi) = \int_{R^2} \phi(\mathbf{k}) \frac{1}{| \mathbf{k} |^2} d\mathbf{k} \tag{26}$$

## 4-RADAR IMAGING WITH WAVELETS

The intent in radar imaging is to obtain a map for the spatial repartition of the target elements contributing to the radar cross-section. In a practical approach this is done by introducing a positive distribution $\mathcal{I}(\mathbf{x}, \mathbf{k})$ such that the integral:

$$S(V) = \int_V \mathcal{I}(\mathbf{x}, \mathbf{k})\, d\mathbf{x} \tag{27}$$

represents the part of the scattering cross-section due to the elements located in the volume V.

Images can as well be associated with backscattering coefficients of one or two variables and, for differentiation, we will use the notation $\mathcal{I}_m(\mathbf{x}, \mathbf{k})$ where $m = 1, 2, 3$ refers to the dimension of the problem.

Each quantity $\mathcal{I}_m(\mathbf{x}, \mathbf{k})$ must have the dimension $(L)^{2-m}$ in order that the l.h.s. of (27) be a surface. Its wavelet expression will be taken under the form ($m = 1, 2, 3$):

$$\mathcal{I}_m(\mathbf{x}, \mathbf{k}) = [K_m(\phi)]^{-1} | C_m(\mathbf{x}, \mathbf{k}) |^2 | \mathbf{k} |^{m-2} \tag{28}$$

where coefficients $K_m$ and $C_m = (H, \phi_{\mathbf{x},\mathbf{k}})_m$ can be obtained from formulas (16)-(26). The interest of definition (28) comes from the general isometric relation (cf(19)) which allows to interpret expressions $C_m/K_m$ as probability densities on the space $(\mathbf{x}, \mathbf{k})$.

The integration of (28) on the whole $\mathbf{x}$-space does not give the real cross-section $| H(\mathbf{k}) |^2$ but only a local mean of this function. For example, the two-dimensional situation leads to the formula:

$$S(\mathbf{k}_0) = \int \mathcal{I}_2(\mathbf{x}_0, \mathbf{k}_0)\, d\mathbf{x}_0$$

$$= [K_2]^{-1} \int | H(k,\theta) |^2 \, k_0^{-2} \, | \phi(\frac{k}{k_0}, \theta - \theta_0) |^2 \ k dk d\theta$$

where notations have been defined for (24).

It can be verified that the mean cross-section transforms as the cross-section itself in a change of observer. This property was in fact ensured by the process of image construction and does not depend on the dimension of the problem.

The above construction is also technically interesting since all computations it introduces can be performed directly from a polar coordinates sampling of the backscattering function $H$. All the implementations are founded on a radial geometric sampling of $H$ and use the fast Mellin transform [8].

## 5-CONCLUDING REMARKS

Dimensional considerations can really have a constructive role in wavelet analysis. This point has been illustrated with the application of the method to radar imaging of laboratory targets.

From a general point of view, the technique leads to the introduction of dependable concepts well inserted in the physical context. The approach could be useful for the study of descriptors beyond their classical domain of use.

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