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ON PERSYMMETRIC COVARIANCE MATRICES IN ADAPTIVE DETECTION.

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Outline of the presentation

I : Introduction.

II : Problem Statement.

III : Statistical Analysis.

IV : Simulations.

V : Conclusions.

Radar detection framework

Principle of radar detection:

- Goal : Detect a known signal $\mathbf{p} \in \mathbb{C}^m$ corrupted by an additive clutter $\mathbf{c}$ by using a binary hypothesis test:

$$\begin{cases} H_0 : \mathbf{y} = \mathbf{c}, & \mathbf{y}_k = \mathbf{c}_k, \text{ for } 1 \leq k \leq K\,, \\ H_1 : \mathbf{y} = A\,\mathbf{p} + \mathbf{c}, & \mathbf{y}_k = \mathbf{c}_k, \text{ for } 1 \leq k \leq K\,, \end{cases} \quad (1)$$

  where $\mathbf{y}$ is the complex m-vector of the received signal,
  A is an unknown complex target amplitude,
  $\mathbf{c}$ is a complex zero-mean Gaussian m-vector with covariance matrix $\mathbf{M} = E[\mathbf{c}\,\mathbf{c}^H]$.
- Under both hypotheses, K signal-free data $\mathbf{y}_k$ are available for clutter parameters estimation. They are called secondary data, independent and identically distributed (i.i.d) with the same distribution as $\mathbf{c}$.

Detection in Gaussian noise: background.

State of art for Gaussian detection:

When **M** is known, the Generalized Likelihood Ratio Test (GLRT) is referred to as the Optimum Gaussian Detector (OGD):

$$\Lambda_{OGD} = \frac{|\mathbf{p}^H \mathbf{M}^{-1} \mathbf{y}|^2}{\mathbf{p}^H \mathbf{M}^{-1} \mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda \,, \tag{2}$$

- The detection threshold λ is related to the PFA by: $\lambda = \sqrt{-\ln(P_{fa})}$.

Problem : in practice, **M** is generally unknown and has to be estimated.

- The Maximum Likelihood theory provides the well-known Sample Covariance Matrix (SCM) built from the secondary data and defined by:

$$\widehat{\mathbf{M}}_{SCM} = \frac{1}{K} \sum_{k=1}^{K} \mathbf{y}_k \, \mathbf{y}_k^H \,. \tag{3}$$

Gaussian noise: adaptive detection (background)

The adaptive Gaussian detector:

Substituting $\widehat{\mathbf{M}}_{SCM}$ for $\mathbf{M}$ in (2) leads to the so-called Adaptive Matched Filter (AMF) test:

$$\Lambda_{AMF} = \frac{|\mathbf{p}^H \, \widehat{\mathbf{M}}_{SCM}^{-1} \, \mathbf{y}|^2}{\mathbf{p}^H \, \widehat{\mathbf{M}}_{SCM}^{-1} \, \mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda \,. \tag{4}$$

- AMF has the Constant False Alarm Rate (CFAR) property.
- Its statistical distribution is known.
- The AMF exhibits a detection loss in comparison with the OGD (for example for the particular choice: $K = 2m \Longrightarrow 3dB$ loss).

Gaussian noise: adaptive detection (background)

$\widehat{\mathbf{M}}_{SCM}$ defined by (3) does not take into account any prior knowledge on the structure of the covariance matrix.

Can we improve its performance with knowledge of prior information on $\widehat{\mathbf{R}}$?

$\Longrightarrow$ Use of a knowledge on the particular structure of the covariance matrix.

Context of this paper

- Many applications can result in a clutter covariance matrix that exhibits some particular structure.

For exemple, radar systems that use a symmetrically spaced linear array for spatial domain processing, or symmetrically spaced pulse train for temporal domain processing.

- In these systems, the clutter covariance matrix $\mathbf{M}$ has the persymmetric property:

$$\mathbf{M} = \mathbf{J}_m \, \mathbf{M}^* \, \mathbf{J}_m \,, \qquad (5)$$

where $\mathbf{J}_m$ is the m-dimensional antidiagonal matrix having 1 as non-zero elements.

- The signal vector is also persymmetric, i.e. it satisfies:

$$\mathbf{p} = \mathbf{J}_m \, \mathbf{p}^* \,. \qquad (6)$$

Contribution of the paper

- The persymmetric structure of $\mathbf{M}$ can be exploited to improve its estimation accuracy compared to the SCM.
- Our study is based on the use in the AMF of the persymmetric Maximum Likelihood (ML) estimate of the clutter covariance matrix in (4) instead of the SCM.
- The distribution of the new detector under hypothesis H_0 can be derived.
- This allows the theoretical detection threshold to be set for a given P_{fa}.

Persymmetry considerations

Let $\mathbf{T}$ be the unitary matrix defined by:

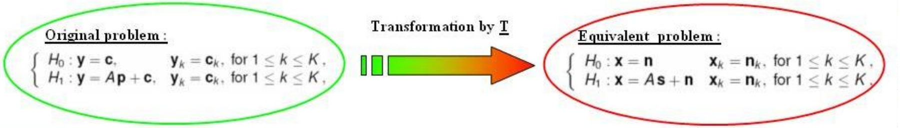
$$\mathbf{T} = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I}_{m/2} & \mathbf{J}_{m/2} \\ i\mathbf{I}_{m/2} & -i\mathbf{J}_{m/2} \end{pmatrix} & \text{for } m \text{ even} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I}_{(m-1)/2} & 0 & \mathbf{J}_{(m-1)/2} \\ 0 & \sqrt{2} & 0 \\ i\mathbf{I}_{(m-1)/2} & 0 & -i\mathbf{J}_{(m-1)/2} \end{pmatrix} & \text{for } m \text{ odd.} \end{cases} \quad (7)$$

Persymmetric vectors and Hermitian matrices are characterized by the following properties:

- $\mathbf{p} \in \mathbb{C}^m$ is a persymmetric vector if and only if $\mathbf{T}\,\mathbf{p}$ is a real vector.
- $\mathbf{M}$ is a persymmetric Hermitian matrix if and only if $\mathbf{T}\,\mathbf{M}\,\mathbf{T}^H$ is a real symmetric matrix.

Equivalent detection problem

Using previous transformation, the original problem can be reformulated as follows:



Equivalent problem transformation.

where $\mathbf{x} \in \mathbb{C}^m$, $\mathbf{n} \sim \mathcal{CN}(0, \mathbf{R})$, $\mathbf{s}$ is a known real vector, **R** is an unknown **real** symmetric matrix. The K transformed secondary data $\mathbf{x}_k$ are i.i.d and share the same $\mathcal{CN}(0, \mathbf{R})$ distribution as $\mathbf{n}$.

The main motivation for introducing the transformed data is that the resulting distribution of the ML estimate of R is very simple.

The ML estimate

- Let us now investigate the ML estimate of the ***real*** covariance matrix $\mathbf{R}$ from the K secondary data $\mathbf{x}_k$.

The ML estimate $\widehat{\mathbf{R}}$ of real matrix $\mathbf{R}$ is unbiased and is given by:

$$\widehat{\mathbf{R}} = \mathcal{R}e(\widehat{\mathbf{R}}_{SCM}) , \quad (8)$$

where $\mathcal{R}e(.)$ stands for the real part, and where:

$$\widehat{\mathbf{R}}_{SCM} = \frac{1}{K} \sum_{k=1}^{K} \mathbf{x}_k \, \mathbf{x}_k^H = \mathbf{T} \, \widehat{\mathbf{M}}_{SCM} \, \mathbf{T}^H . \quad (9)$$

Remark that $K \, \widehat{\mathbf{R}}$ is real Wishart distributed with $2K$ degrees of freedom with parameter matrix $\frac{1}{2} \, \mathbf{R}$.

Optimized adaptive detector : The PAMF

Using $\widehat{\mathbf{R}}$ in the AMF leads to the following detection test, called the P-AMF,

$$\Lambda_{PAMF} = \frac{|\mathbf{s}^{\top}\widehat{\mathbf{R}}^{-1}\mathbf{x}|^2}{\mathbf{s}^{\top}\widehat{\mathbf{R}}^{-1}\mathbf{s}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda\,, \tag{10}$$

or equivalently, in terms of the original data,

$$\Lambda_{PAMF} = \frac{|\mathbf{p}^H\mathbf{T}^H[\mathcal{R}e(\mathbf{T}\widehat{\mathbf{M}}_{SCM}\mathbf{T}^H)]^{-1}\mathbf{T}\mathbf{y}|^2}{\mathbf{p}^H\mathbf{T}^H[\mathcal{R}e(\mathbf{T}\widehat{\mathbf{M}}_{SCM}\mathbf{T}^H)]^{-1}\mathbf{T}\mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda\,. \tag{11}$$

- Taking into account the real structure of **R** (or equivalently the persymmetric structure of **M**) in the ML estimation procedure virtually doubles the amount of secondary data.
- Performance of this new detector has to be studied.

Statistical study of the PAMF

Probability Density Function (PDF) of this detector.

A statistical study allows to determine under H_0, the PDF of Λ_{PAMF} defined by (10):

$$p(z) = \frac{(2K-m+1)\,(2K-m+2)}{2\,K\,(2K+1)} \times {}_2F_1\left(\frac{2K-m+3}{2}, \frac{2K-m+4}{2}, \frac{2K+3}{2}; -\frac{z}{K}\right), \tag{12}$$

where ${}_2F_1$ is the hypergeometric function.

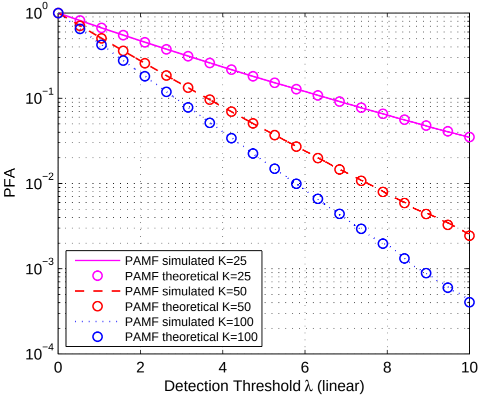
Statistical study of the PAMF

Relationship between PFA and the detection threshold.

By using previous relation, the relationship between the PFA and the detection threshold λ is:

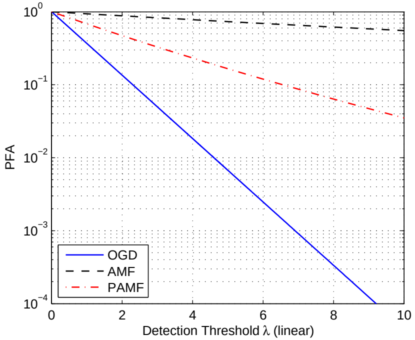
$$P_{fa} = {}_2F_1\left(\frac{2K-m+1}{2}, \frac{2K-m+2}{2}, \frac{2K+1}{2}; -\frac{\lambda}{K}\right).$$

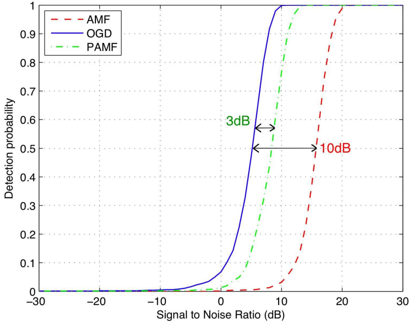
PFA vs the P-AMF detection threshold



These plots show the perfect agreement between the theory (circles) and the Monte-Carlo trials (solid lines) for different values of K and $m = 20$.

PD vs SNR relation





- Left figure: Threshold decreasing brought by the P-AMF compared to the AMF for $K = 25$ and $m = 20$.
- Right figure: Improvement of about 7dB in terms of detection for the PAMF compared to the AMF for this set of parameters.

Conclusions.

- A new adaptive detection test which takes into account the persymmetric structure of the clutter covariance matrix.
- Estimation by a Maximum Likelihood procedure.
- Derivation of the analytical distribution of the P-AMF test statistic. This result allows the detection threshold to be set for a given PFA.
- Simulations validate theoretical results and show significant improvement in the detection performance of PAMF over the conventional AMF, especially for a small number of secondary data ($K < 2m$).