Derivation of the Bias of the Normalized Sample Covariance Matrix in a Heterogeneous Noise With Application to Low Rank STAP Filter

Guillaume Ginolhac, Philippe Forster, Frédéric Pascal, and Jean-Philippe Ovarlez

***Abstract*—In a previous work, we have developed a low-rank (LR) spatio-temporal adaptive processing (STAP) filter when the disturbance is modeled as the sum of a low-rank spherically invariant random vector (SIRV) clutter and a zero-mean white Gaussian noise. This LR-STAP filter is built from the normalized sample covariance matrix (NSCM) and exhibits good robustness properties to secondary data contamination by target components. In this correspondence, we derive the bias of the NSCM with this noise model. We show that the eigenvectors estimated from the NSCM are unbiased. The new expressions of the expectation of NSCM eigenvalues are also given. From these results, we also show that the estimate of the clutter subspace projector based on the NSCM used in our LR-STAP is a consistent estimate of the true one. Results on numerical data validates the theoretical approach.**

***Index Terms*—Bias study, consistency, low rank clutter, normalized sample covariance matrix, SIRV, orthogonal projector, space time adaptive processing (STAP).**

I. Introduction

Space time adaptive processing (STAP) is a technique used in airborne phased array radar to detect moving target embedded in an interference background such as jamming or strong clutter [1]. While conventional radars are capable of detecting targets both in the time domain related to target range and in the frequency domain related to target velocity, STAP uses an additional domain (space) related to the target angular localization. The consequence is a two-dimensional adaptive filtering technique which uses jointly temporal and spatial dimensions to suppress interferences and to improve target detection. In most works on radar, the clutter is assumed to be a lonely Gaussian process. However, the increase of the radar resolution leads to a higher scene heterogeneity where the clutter can no longer be modeled by a Gaussian process [2]–[4]. To take this heterogeneity into account, one can use the spherically invariant random vector (SIRV) product model, first introduced by Yao [5] for the information theory. This is a compound-Gaussian model, well-known for its good statistical properties as well as it has been validated on several real data sets [6]–[9]. Moreover, in some geometrical STAP configurations (e.g., side looking), the clutter can be considered to have a low rank structure (Brennan's rule [10]). Therefore, the disturbance in this correspondence is modeled as the sum of a LR-SIRV clutter and a zero-mean white Gaussian noise. It is a general model as considered in [11].

In practice, the disturbance covariance matrix is generally unknown and an estimate is required to perform the STAP processing. In the context of LR clutter, only the estimation of the clutter subspace projector is needed for the processing [12], [13]. This estimation procedure requires the so-called secondary data. These data are assumed to be independent and share the same distribution as the observation under test. In a STAP framework, the dimension of the covariance matrix is important (number of sensors times number of pulses). Commonly, the number of secondary data has to be upper than two times this dimension to ensure the classical 3-dB loss for the performance results. One of the advantage of the LR techniques is that this rule can be strongly relaxed to preserve such a performance [12], [13]. Another problem in STAP comes from these secondary data which are often contaminated by the secondary lobes of the target under study or other targets with same angular and velocity properties. It is also possible to have outliers in the STAP data cube like for example in highway STAP data with the cars traffic-jam or the convoys tracking. In this case, LR techniques exhibit another advantage on classical methods: for a quite low signal-to-clutter ratio (SCR), LR techniques are robust to secondary data contamination, see e.g., [12], [14]. This robustness is directly relied on the choice of the covariance matrix estimate, from which the projector estimate will be derived. It seems obvious that the well-known sample covariance matrix (SCM) is not adapted to strong contamination problems since this estimate depends on the power of each data. Thus, it is interesting to consider other covariance matrix estimates, independent to the data power, i.e., self-normalized estimates. This is the case of the normalized sample covariance matrix (NSCM) [15] and the fixed point estimate (FPE) [16], [17]. Even if the FPE exhibits better statistical properties than the NSCM [18] (NSCM is a biased estimate of the covariance matrix) and provides an adaptive detector with the SIRV-CFAR property (texture and covariance matrix CFAR) [19], the FPE is not adapted to LR context (it requires an inversion of matrix, which is impossible in the case of a small number of secondary data). We then proposed in [20] to build the clutter subspace projector based on the NSCM. The corresponding LR STAP filter exhibits interesting results: it is more robust to secondary data contamination by target components than the LR STAP filter obtained from the SCM. In this correspondence, we are interested in the statistical properties of the NSCM and the corresponding projector in a LR-SIRV clutter plus white Gaussian noise context. This study was performed in [21] in an only SIRV noise context. Unfortunately, this noise context is not realistic for STAP system. In this correspondence, this property is extended for the LR-SIRV plus white Gaussian noise case and the new expressions of the expectation of NSCM eigenvalues are given. From these results, we also show that the estimate of the clutter subspace projector based on the NSCM is a consistent estimate of the true one. Main difficulty in the LR context is that the texture of the SIRV has to be taken into account. After the theoretical approach, validation on simulated data are proposed to enlighten our results.

The correspondence is organized as follows. Section II presents the considered STAP problem. In Section III, the theoretical study on the NSCM is presented. Section IV shows results on simulated data. The following convention is adopted: italic indicates a scalar quantity, lower case boldface indicates a vector quantity and upper case boldface a matrix. ^{T} denotes the transpose operator and ^{H} the transpose conjugate. $E[\ ]$ is the expected value operator and $\|\|_F$ is the Frobenius norm. $\mathcal{CN}(\mathbf{a}, \mathbf{M})$ is a complex Gaussian vector of mean $\mathbf{a}$ and of covariance matrix $\mathbf{M}$.

II. Problem Statement and Low-Rank STAP Filters

STAP [1] is applied to airborne radar in order to detect moving targets. Typically, the radar receiver consists in an array of N antenna elements processing M pulses in a coherent processing interval. In this framework, the problem is to detect a complex signal $\mathbf{s}$ corrupted by an additive disturbance $\mathbf{d}$ in an observation $\mathbf{y} \in \mathbb{C}^{NM}$:

$$\begin{aligned} \mathbf{y} &= \mathbf{s} + \mathbf{d} \\ \mathbf{y}_i &= \mathbf{d}_i \quad i = 1, \ldots, N_s \end{aligned} \tag{1}$$

where the $\mathbf{y}_i$'s $\in \mathbb{C}^{NM}$ are the N_s secondary data which are assumed signal-free independent measurements. $\mathbf{y}$ is referred to as the primary

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data. Disturbances $\mathbf{d}$ and $\mathbf{d}_i$ are assumed to be the sum of a clutter, $\mathbf{c}$ or $\mathbf{c}_i$, and a Gaussian white noise, $\mathbf{n}$ or $\mathbf{n}_i$:

$$\begin{aligned}\mathbf{d} &= \mathbf{c} + \mathbf{n} \\ \mathbf{d}_i &= \mathbf{c}_i + \mathbf{n}_i \quad i = 1, \ldots, N_s.\end{aligned} \tag{2}$$

We assume that $\mathbf{d}$ and $\mathbf{d}_i$ are independent and share the same statistical distribution.

The processes $\mathbf{n}$ and $\mathbf{n}_i$ are modeled as a zero-mean complex Gaussian noise $\mathcal{CN}(\mathbf{0}, \lambda \mathbf{I}_{NM})$. Concerning the clutter $\mathbf{c}$ and $\mathbf{c}_i$, a non-homogeneous case will be considered as in [11]. In this case, the clutter power in each cell i and the cell under test is different. In such a situation, it is common to model this kind of clutter by a SIRV [5] which is a non-homogeneous Gaussian random vector with random power. The SIRV $\mathbf{c}$ (respectively, $\mathbf{c}_i$) is the product of a positive random variable τ (respectively, τ_i), called the *texture*, and a NM-dimensional independent complex Gaussian vector $\mathcal{CN}(\mathbf{0}, \mathbf{M})$ $\mathbf{g}$ (respectively, $\mathbf{g}_i$), called the *speckle*, with zero-mean and covariance matrix $\mathbf{M} = E(\mathbf{g}\mathbf{g}^H) = E(\mathbf{g}_i \mathbf{g}_i^H)$:

$$\begin{aligned}\mathbf{c} &= \sqrt{\tau}\mathbf{g} \\ \mathbf{c}_i &= \sqrt{\tau}_i \mathbf{g}_i \quad i = 1, \ldots, N_s.\end{aligned} \tag{3}$$

Note that the covariance matrix is normalized according to $\mathrm{Tr}(\mathbf{M}) = NM$ (see, e.g., [16]) for identifiability considerations. The covariance matrix of $\mathbf{d}$ and $\mathbf{d}_i$ is then given by

$$\boldsymbol{\Sigma} = \bar{\tau}\mathbf{M} + \lambda \mathbf{I}_{NM} \tag{4}$$

where $\bar{\tau} = E(\tau)$. In the STAP context, we are able to evaluate the clutter rank thanks to the Brennan's Rule [10], $\mathrm{rank}(\mathbf{M}) = r \ll NM$, which leads to a low rank structure for the STAP clutter $\mathbf{c}$ and $\mathbf{c}_i$. The *speckle* covariance matrix $\mathbf{M}$ can be thus decomposed as

$$\mathbf{M} = \sum_{k=1}^{r} \lambda_k \mathbf{u}_k \mathbf{u}_k^H \tag{5}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_r > \lambda_{r+1} = \cdots = \lambda_{NM} = 0$ are the eigenvalues of $\mathbf{M}$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_{NM}\}$ are the associated eigenvalues. Therefore, we rewrite the expression of the covariance matrix of the disturbance

$$\boldsymbol{\Sigma} = \sum_{k=1}^{r} \sigma_k \mathbf{u}_k \mathbf{u}_k^H + \lambda \sum_{k=r+1}^{NM} \mathbf{u}_k \mathbf{u}_k^H \tag{6}$$

where $\sigma_1 = \bar{\tau}\lambda_1 + \lambda > \sigma_2 = \bar{\tau}\lambda_2 + \lambda > \cdots > \sigma_r = \bar{\tau}\lambda_r + \lambda > \lambda = \cdots = \lambda$ are the eigenvalues of $\boldsymbol{\Sigma}$.

Let us define the two following unitary matrices $\mathbf{U}_r = [\mathbf{u}_1 \ldots \mathbf{u}_r]$ and $\mathbf{U}_0 = [\mathbf{u}_{r+1} \ldots \mathbf{u}_{NM}]$ which allows to build the projector onto the clutter subspace $\boldsymbol{\Pi}_c$ and the projector onto the orthogonal of the clutter subspace $\boldsymbol{\Pi}_c^{\perp}$ [12], [13]:

$$\begin{aligned}\boldsymbol{\Pi}_c &= \mathbf{U}_r \mathbf{U}_r^H = \sum_{k=1}^{r} \mathbf{u}_k \mathbf{u}_k^H \\ \boldsymbol{\Pi}_c^{\perp} &= \mathbf{U}_0 \mathbf{U}_0^H = \mathbf{I}_{NM} - \boldsymbol{\Pi}_c = \sum_{k=r+1}^{NM} \mathbf{u}_k \mathbf{u}_k^H\end{aligned} \tag{7}$$

Up to a constant factor, the optimal STAP filter $\mathbf{w}_{\mathrm{opt}}$ is written as follows [1]:

$$\mathbf{w}_{\mathrm{opt}} = \boldsymbol{\Sigma}^{-1}\mathbf{s}. \tag{8}$$

In low-rank assumption, this optimal filter is [12], [13]

$$\mathbf{w}_{\mathrm{lr-opt}} = \boldsymbol{\Pi}_c^{\perp}\mathbf{s}. \tag{9}$$

In practical cases, the projector onto clutter subspace, $\boldsymbol{\Pi}_c$, is unknown and has to be estimated from the secondary data $\mathbf{y}_i$ of (1) in order to provide adaptive versions of the above STAP filters.

III. Theoretical Study of the Bias of the NSCM

For the case of a Gaussian clutter plus a white Gaussian noise, the well-known method to obtain the projector onto the clutter subspace, $\boldsymbol{\Pi}_c$, is based on the SCM [12], [13] which is the common way to estimate the covariance matrix $\boldsymbol{\Sigma}$. For the case of a LR-SIRV clutter plus a white Gaussian noise, we proposed in [20] to build the projector onto the clutter subspace, $\boldsymbol{\Pi}_c$, from the NSCM. The NSCM was first introduced in [15] to deal with detection in non-homogeneous clutter. The NSCM, $\hat{\mathbf{R}}_{\mathrm{NSCM}}$, is defined by

$$\hat{\mathbf{R}}_{\mathrm{NSCM}} = \frac{NM}{N_s} \sum_{i=1}^{N_s} \frac{\mathbf{y}_i \mathbf{y}_i^H}{\mathbf{y}_i^H \mathbf{y}_i}. \tag{10}$$

Let us recall the steps to obtain the estimate of $\boldsymbol{\Pi}_c$ from the NSCM. The eigenvalue decomposition of $\hat{\mathbf{R}}_{\mathrm{NSCM}}$ is first performed:

$$\hat{\mathbf{R}}_{\mathrm{NSCM}} = \sum_{k=1}^{r} \hat{\mu}_k \hat{\mathbf{u}}_k \hat{\mathbf{u}}_k^H + \sum_{k=r+1}^{NM} \hat{\mu}_k \hat{\mathbf{u}}_k \hat{\mathbf{u}}_k^H. \tag{11}$$

Let us define the two unitary matrices $\hat{\mathbf{U}}_r = [\hat{\mathbf{u}}_1 \ldots \hat{\mathbf{u}}_r]$ and $\hat{\mathbf{U}}_0 = [\hat{\mathbf{u}}_{r+1} \ldots \hat{\mathbf{u}}_{NM}]$. The estimated projector onto the clutter subspace [12], [13] is

$$\begin{aligned}\hat{\boldsymbol{\Pi}}_{c\mathrm{NSCM}} &= \hat{\mathbf{U}}_r \hat{\mathbf{U}}_r^H = \sum_{k=1}^{r} \hat{\mathbf{u}}_k \hat{\mathbf{u}}_k^H \\ \hat{\boldsymbol{\Pi}}_{c\mathrm{NSCM}}^{\perp} &= \hat{\mathbf{U}}_0 \hat{\mathbf{U}}_0^H = \mathbf{I}_{NM} - \hat{\boldsymbol{\Pi}}_{c\mathrm{NSCM}} = \sum_{k=r+1}^{NM} \hat{\mathbf{u}}_k \hat{\mathbf{u}}_k^H.\end{aligned} \tag{12}$$

Let $\mathbf{R} = E[\hat{\mathbf{R}}_{\mathrm{NSCM}}]$ be the expectation of the NSCM. The eigenvectors and the eigenvalues of $\mathbf{R}$ are denoted by $\mathbf{v}_k$ and μ_k with $k = 1, \ldots, NM$. It was shown in [21] that, despite the bias of the NSCM, $\mathbf{R}$ has the same eigenvectors as the true covariance matrix when the disturbance is only composed of a SIRV noise. The associated eigenvalues are different, but their ordering and their multiplicity are identical. In the proof of this result, the texture plays no role because it is canceled in the normalization of the NSCM. However, the texture must be taken into account for the study of the bias of the NSCM in the LR-SIRV clutter plus white Gaussian noise context of this correspondence. This is the purpose of the following proposal.

Proposition III.1:

1) $\mathbf{R}$ and $\boldsymbol{\Sigma}$ share the same eigenvectors ($\mathbf{v}_k = \mathbf{u}_k$ for $k = 1, \ldots, NM$) with the same ordering and multiplicity of the corresponding eigenvalues (expressions of eigenvalues of $\mathbf{R}$ are obtained and given in (20)).
2) $\hat{\boldsymbol{\Pi}}_{c\mathrm{NSCM}}$ is a consistent estimate of $\boldsymbol{\Pi}_c$.

Proof: We have

$$\mathbf{R} = E\left[\frac{\mathbf{y}_i \mathbf{y}_i^H}{\mathbf{y}_i^H \mathbf{y}_i}\right] = E\left[\frac{\mathbf{d}_i \mathbf{d}_i^H}{\mathbf{d}_i^H \mathbf{d}_i}\right]. \tag{13}$$

The vector $\mathbf{y}_i = \mathbf{d}_i$ is, conditionally to τ, a complex Gaussian vector with covariance matrix

$$\boldsymbol{\Sigma}_\tau = \tau\mathbf{M} + \lambda \mathbf{I}_{NM} = \sum_{k=1}^{r} \sigma_k^\tau \mathbf{u}_k \mathbf{u}_k^H + \sigma^\tau \sum_{k=r+1}^{NM} \mathbf{u}_k \mathbf{u}_k^H \tag{14}$$

where $\sigma_1^\tau = \tau\lambda_1 + \lambda > \sigma_2^\tau = \tau\lambda_2 + \lambda > \cdots > \sigma_r^\tau = \tau\lambda_r + \lambda > \sigma^\tau = \lambda \ldots \sigma^\tau = \lambda$ are the eigenvalues of $\boldsymbol{\Sigma}_\tau$. We have computed in [21] the eigenvalues and the eigenvectors of a normalized Gaussian vector. Let $\mathbf{R}_\tau$ be the covariance matrix of $(\mathbf{d}_i)/(\|\mathbf{d}_i\|)$ conditionally to τ. Results of [21] show that

$$\mathbf{R}_\tau = E\left[\frac{\mathbf{d}_i\,\mathbf{d}_i^H}{\mathbf{d}_i^H\,\mathbf{d}_i}\middle|\tau\right] = \sum_{k=1}^{r}\mu_k^\tau \mathbf{u}_k\mathbf{u}_k^H + \mu^\tau \sum_{k=r+1}^{NM}\mathbf{u}_k\mathbf{u}_k^H \tag{15}$$

where $\mu_1^\tau > \mu_2^\tau > \cdots > \mu_r^\tau > \mu^\tau = \cdots = \mu^\tau$ are the eigenvalues of $\mathbf{R}_\tau$ given by

$$\mu_k^\tau = E\left[\frac{NM(\tau\lambda_k+\lambda)\chi_k(2)}{\sum_{j=1}^{r}(\tau\lambda_j+\lambda)\chi_j(2)+\lambda\chi_{r+1}(2(NM-r))}\middle|\tau\right]$$
$$\mu^\tau = E\left[\frac{NM\lambda\chi_{r+2}(2)}{\sum_{j=1}^{r}(\tau\lambda_j+\lambda)\chi_j(2)+\lambda\chi_{r+2}(2)+\lambda\chi_{r+3}(2(NM-r-1))}\middle|\tau\right] \tag{16}$$

where $\chi_m(n)$ is a Chi-squared distributed random variable with n degrees of freedom. All the random variables $\chi_m(n)$ are independent.

It is next easy to deduce the covariance matrix, $\mathbf{R}$, of the vectors $\frac{\mathbf{d}_i}{\|\mathbf{d}_i\|}$:

$$\mathbf{R} = E[\mathbf{R}_\tau] = E\left[\frac{\mathbf{d}_i\,\mathbf{d}_i^H}{\mathbf{d}_i^H\,\mathbf{d}_i}\right] \tag{17}$$

where the expectation E is here taken over τ. From (15), we have

$$\mathbf{R} = \sum_{k=1}^{r}\mu_k \mathbf{u}_k\mathbf{u}_k^H + \mu \sum_{k=r+1}^{NM}\mathbf{u}_k\mathbf{u}_k^H \tag{18}$$

where $\mu_k = E[\mu_k^\tau]$, $\mu = E[\mu^\tau]$ (E is still taken over τ) are the eigenvalues of $\mathbf{R}$ and

$$\mu_1 > \mu_2 > \cdots > \mu_r > \mu = \cdots = \mu. \tag{19}$$

The expressions of μ_k and μ follow from (16):

$$\mu_k = E\left[\frac{NM(\tau\lambda_k+\lambda)\chi_k(2)}{\sum_{j=1}^{r}(\tau\lambda_j+\lambda)\chi_j(2)+\lambda\chi_{r+1}(2(NM-r))}\right]$$
$$\mu = E\left[\frac{NM\lambda\chi_{r+2}(2)}{\sum_{j=1}^{r}(\tau\lambda_j+\lambda)\chi_j(2)+\lambda\chi_{r+2}(2)+\lambda\chi_{r+3}(2(NM-r-1))}\right] \tag{20}$$

where expectations are taken over τ. Therefore, we have the proof of the first point of Proposal III.1.

From the law of large numbers, we have

$$\hat{\mathbf{R}}_{\text{NSCM}} \xrightarrow{P} \mathbf{R} \text{ when } N_s \longrightarrow \infty. \tag{21}$$

Let f be the function $f(\mathbf{A})$ which associates to a Hermitian matrix $\mathbf{A}$ the projector on the eigenvectors associated to the r greatest eigenvalues. As this function is continuous at $\mathbf{A} = \mathbf{R}$, one has from a corollary of LLN:

$$f(\hat{\mathbf{R}}_{\text{NSCM}}) = \hat{\mathbf{\Pi}}_{c\text{NSCM}} \xrightarrow{P} f(\mathbf{R}) = \mathbf{\Pi}_c \text{ when } N_s \longrightarrow \infty \tag{22}$$

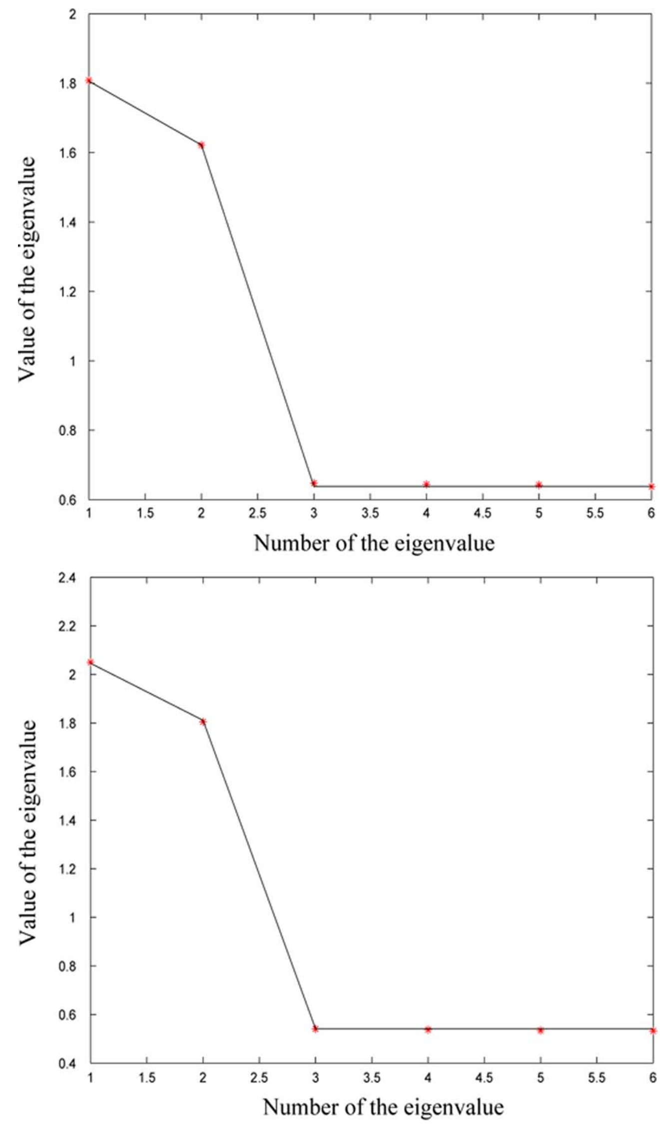

Fig. 1. Theoretical eigenvalues μ_k and μ (black solid line) and experimental eigenvalues $\mu_{\exp}$ (red star) for $\nu = 0.5$ (left) and $\nu = 2$ (right).

which shows the last point of III.1, namely the consistency of the estimate $\hat{\mathbf{\Pi}}_{c\text{NSCM}}$. ■

IV. Simulation for Consistency

In this section, we check the theoretical expressions of the expectation of NSCM eigenvalues, μ_k and μ, given by (20) and the consistency of the estimate $\hat{\mathbf{\Pi}}_{c\text{NSCM}}$.

Let $\mathbf{e}_k$ be the k's vector of the canonical basis of $\mathbb{C}^{NM}$. Without loss of generality for our purpose, we will assume that $\mathbf{M}$ in (5) is given by

$$\mathbf{M} = \sum_{k=1}^{r}\lambda_k \mathbf{e}_k\mathbf{e}_k^H. \tag{23}$$

In this simple case, the projector $\mathbf{\Pi}_c$ onto the clutter subspace is equal to

$$\mathbf{\Pi}_c = \sum_{k=1}^{r}\mathbf{e}_k\mathbf{e}_k^H. \tag{24}$$

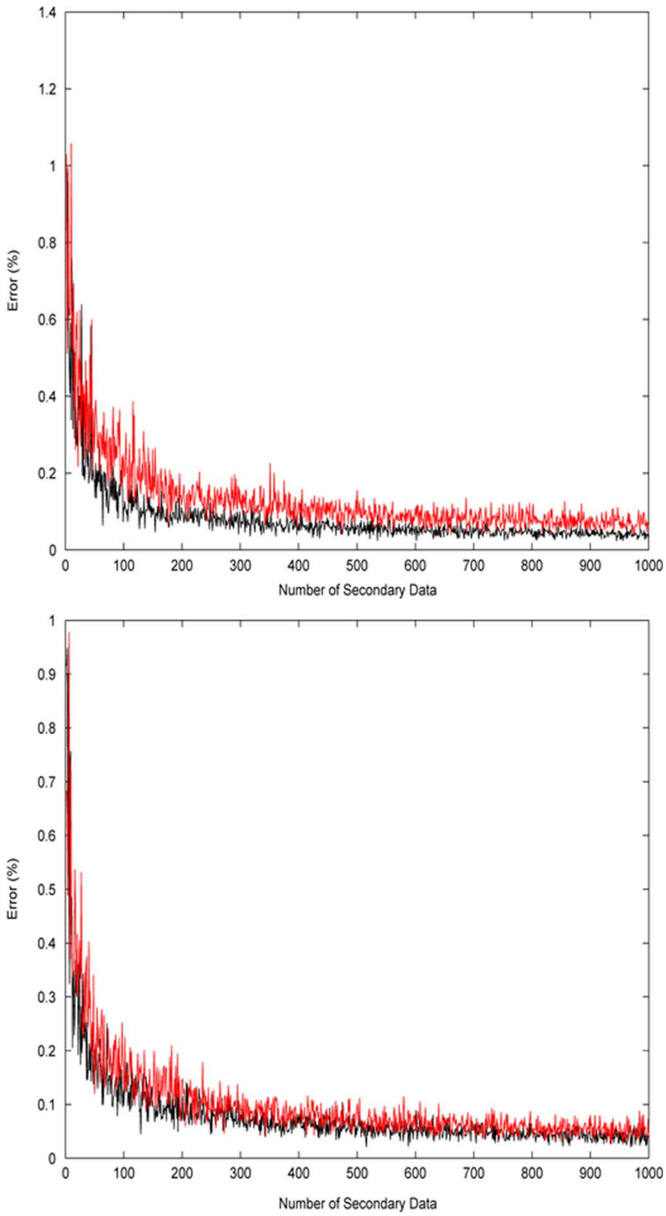

Fig. 2. E_{SCM} (dark) and E_{NSCM} (red) as a function of the number N_s of secondary data for $\nu = 0.5$ (left) and $\nu = 2$ (right).

Concerning the N_s secondary data $\{\mathbf{y}_i\}_{1,N_s}$

$$\mathbf{y}_i = \sqrt{\tau_i}\mathbf{g}_i + \mathbf{n}_i, \tag{25}$$

$\mathbf{n}_i$ is $\mathcal{CN}(\mathbf{0}, \lambda\mathbf{I}_{NM})$ distributed, $\mathbf{g}_i$ is $\mathcal{CN}(\mathbf{0}, \mathbf{M})$ distributed (with $\mathbf{M}$ defined in (23)) and τ_i is Gamma distributed with shape parameter ν and scale parameter $1/\nu$ (which results in $E[\tau_i] = 1$). We have the following values for the previous parameters: $NM = 6$ and $r = 2$, $\lambda_1 = 5$, $\lambda_2 = 4$ and $\lambda = 1$, and $\nu = 0.5$ to simulate a strong non-homogeneous clutter and $\nu = 2$ to simulate a quasi-homogeneous clutter. The clutter-to-noise ratio (CNR) is fixed to 25 dB.

We first evaluate from 10 000 Monte Carlo trials the expressions of μ_k and μ, given in (20).

The expectation of the NSCM defined in (10) is next estimated by averaging 10 000 trials, each of them using $N_s = 10$ secondary data of the form of (25). From this average matrix, the eigenvalues, denoted $\mu_{\exp}$, are computed.

Fig. 1 shows the theoretical eigenvalues μ_k and μ and the experimental ones $\mu_{\exp}$ for $\nu = 0.5$ and $\nu = 2$. We notice good match between these results, which validates the expressions of (20).

Now, we check the consistency of $\hat{\mathbf{\Pi}}_{c\mathrm{NSCM}}$ stated in (22) when $N_s \longrightarrow \infty$. For comparison, we also illustrate the consistency of $\hat{\mathbf{\Pi}}_{c\mathrm{SCM}}$.

We build the SCM and the NSCM defined in (10) from the secondary data of (25). In this simulation, the number of secondary data N_s increases from 2 to 1000. For each number N_s of secondary data, the two estimates of the projector onto the clutter subspace, $\hat{\mathbf{\Pi}}_{c\mathrm{SCM}}(N_s)$ and $\hat{\mathbf{\Pi}}_{c\mathrm{NSCM}}(N_s)$ (12) are obtained. Finally, the two following relative errors are computed:

$$E_{\mathrm{SCM}}(N_s) = \frac{\|\hat{\mathbf{\Pi}}_{c\mathrm{SCM}}(N_s) - \mathbf{\Pi}_c\|_F}{\|\mathbf{\Pi}_c\|_F}$$
$$E_{\mathrm{NSCM}}(N_s) = \frac{\|\hat{\mathbf{\Pi}}_{c\mathrm{NSCM}}(N_s) - \mathbf{\Pi}_c\|_F}{\|\mathbf{\Pi}_c\|_F}. \tag{26}$$

Fig. 2 shows E_{SCM} and E_{NSCM} as a function of N_s for $\nu = 0.5$ and $\nu = 2$. Both errors are very small (less than 1%) and decrease to 0 when $N_s \longrightarrow \infty$. This result illustrates the consistency of $\hat{\mathbf{\Pi}}_{c\mathrm{NSCM}}$.

V. Conclusion

In this correspondence, we showed, in a LR-SIRV clutter plus white Gaussian noise context, that the estimated eigenvectors of the NSCM are unbiased. The expressions of the expectation of NSCM eigenvalues were also given. From these results, we demonstrated that the estimation of the projector onto the clutter subspace obtained from the NSCM is a consistent estimate of the true one. Results on numerical data validated the theoretical approach. As shown in [20], this new estimate allows to build a LR STAP filter which allows a better robustness to secondary data contamination by target components than the one obtained from the SCM. In future work, the theoretical performances of this new LR STAP filter built from the NSCM will be computed by using results obtained in this correspondence.

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On the Achievability of Cramér–Rao Bound in Noisy Compressed Sensing

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Abstract*—Recently, it has been proved in Babadi *et al.* [B. Babadi, N. Kalouptsidis, and V. Tarokh, "Asymptotic achievability of the Cramér–Rao bound for noisy compressive sampling," *IEEE Trans. Signal Process.*, vol. 57, no. 3, pp. 1233–1236, 2009] that in noisy compressed sensing, a joint typical estimator can asymptotically achieve the Cramér–Rao lower bound of the problem. To prove this result, Babadi *et al.* used a lemma, which is provided in Akçakaya and Tarokh [M. Akçakaya and V. Trarokh, "Shannon theoretic limits on noisy compressive sampling," *IEEE Trans. Inf. Theory*, vol. 56, no. 1, pp. 492–504, 2010] that comprises the main building block of the proof. This lemma is based on the assumption of Gaussianity of the measurement matrix and its randomness in the domain of noise. In this correspondence, we generalize the results obtained in Babadi *et al.* by dropping the Gaussianity assumption on the measurement matrix. In fact, by considering the measurement matrix as a deterministic matrix in our analysis, we find a theorem similar to the main theorem of Babadi *et al.* for a family of randomly generated (but deterministic in the noise domain) measurement matrices that satisfy a generalized condition known as "the concentration of measures inequality." By this, we finally show that under our generalized assumptions, the Cramér–Rao bound of the estimation is achievable by using the typical estimator introduced in Babadi *et al.

***Index Terms*—Chernoff bound, compressed sensing, joint typicality, typical estimation.**

I. Introduction

Compressed sensing (CS), which is also known as compressive sampling [3]–[5], is a well-known method for taking linear measurements from a sparse vector. Compressed sensing proposes that one can recover a sparse signal from a few number of measurements, and so it can override the usual sampling method based on Nyquist criteria [3]. In this correspondence, we revisit the problem of signal recovery in noisy compressed sensing, in which the above mentioned measurements are blended with noise. Indeed, suppose that noisy measurements of the sparse signal are taken by a random measurement matrix in the following form:

$$\mathbf{y} = \mathbf{A}\mathbf{s} + \mathbf{n}, \tag{1}$$

in which $\mathbf{s}$ is the original $M \times 1$ sparse signal, $\mathbf{y}$ is the $N \times 1$ vector of measurements, $\mathbf{n} \sim N(0, \sigma_n^2 I_{N\times N})$ is an $N \times 1$ Gaussian noise vector and $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_M]$ is an $N \times M$ measurement matrix whose elements are usually generated at random. More precisely, these elements are independent and identically distributed random variables

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